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STATICALLY TAME PERIODIC HOMEOMORPHISMS OF COMPACT CONNECTED 3-MANIFOLDS. II. STATICALLY TAME IMPLIES TAME

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ABSTRACT. Let f be a periodic homeomorphism $M \leftrightarrow M$, where M is a compact connected 3-manifold (without boundary). Suppose that for each i, the fixed-point set of f^i is a tame set. Then f is simplicial, relative to some triangulation of M.

1. Statement of results. Let M be a 3-manifold (without boundary), and let K be a triangulation of M. A set $L \subset M$ is *tame* (relative to K) if there is a homeomorphism h: $M \leftrightarrow M$ such that $h(L)$ is a polyhedron (relative to K). By the Hauptvermutung, this condition is independent of the choice of K. Hence we define a *tame* set in M to be a set which is tame relative to some triangulation of M.

Let f: $M \leftrightarrow M$ be a homeomorphism of period n. For each i, let F_i be the set of all fixed points of f^i. If each set F_i is tame, then f is *statically tame*. If f is simplicial, relative to some triangulation of M, then f is *tame*. It is trivial to observe that tameness for f implies static tameness. If f is simplicial relative to K, then f is simplicial relative to the first barycentric subdivision bK; and if $\sigma \in$ bK, then $f^i(\sigma) = \sigma$ only if $f^i|\sigma$ is the identity. Thus each set F_i forms a subcomplex of bK.

Let $n = p_1 p_2 \cdot \cdot \cdot p_r$, where the p_i's are primes, and $p_i \leqslant p_{i+1}$ for $i < r$. For each $i \leqslant r$, let $n_i = n/p_i$. It turns out that in dealing with statically tame homeomorphisms, all that we really need to know is that each set F_{n_i} is tame. If the latter condition holds, then we say that f is *weakly statically tame*.

THEOREM 1.1. *If* (1) *M is compact and connected,* (2) *f is weakly statically tame, and* (3) *no point of $\bigcup_i F_{n_i}$ is isolated, then* (4) *f is tame.*

After the proof of Theorem 1.1 was written, there appeared a paper of J. H. Rubinstein **[R]** presenting a proof of the following.

THEOREM 1.2 (J. H. RUBINSTEIN). *Let N be a compact connected nonorientable 3-manifold with boundary, such that the boundary* Bd N *is the union of two projective planes; and suppose that $\pi(N) \approx \mathbf{Z}_2$. Let $\tilde{N}$ be a connected 2-fold orientable covering of N, and suppose that $\tilde{N}$ is homeomorphic to the product $S^2 \times I$ of a 2-sphere with a*

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closed linear interval. Then there is an annulus A in N such that (1) $\operatorname{Bd} A = A \cap \operatorname{Bd} N$, (2) *the two components of* $\operatorname{Bd} A$ *lie in different components of* $\operatorname{Bd} N$, *and* (3) *neither component of* $\operatorname{Bd} N$ *is contractible in N.*

Rubinstein remarks that as in an earlier paper of G. R. Livesay **[L]** it follows that N is homeomorphic to a product $P^2 \times I$, where P^2 is a projective plane. This fills the gap in the proof of Lemma 3.1 of a paper of Morris W. Hirsch and Stephen Smale; see pp. 898–899 of **[H-S]**; and in **[H-S]** it was shown that Lemma 3.1 implies the following.

THEOREM 1.3 (HIRSCH-SMALE). *Let M be a 3-manifold, let f be an involution $M \leftrightarrow M$, and let P be an isolated fixed point of f. Then P has a 3-cell neighborhood C^3 such that $f(C^3) = C^3$ and $f|C^3$ is equivalent to a linear involution.*

That is, there is a homeomorphism h: $C^3 \leftrightarrow \mathbf{B}^3$, of C^3 onto the unit ball $\mathbf{B}^3$ in $\mathbf{R}^3$, such that $f = h^{-1}rh$, where $r(x, y, z) = (-x, -y, -z)$. Hirsch and Smale also showed (modulo their Lemma 3.1) that if C^3 is as in Theorem 1.3, and h_0 is a homeomorphism $\operatorname{Bd} C^3 \leftrightarrow \operatorname{Bd} \mathbf{B}^3$, then h_0 can be extended so as to give an h as in Theorem 1.3.

Using Theorem 1.3, we can omit the ad hoc hypothesis in Theorem 1.1, obtaining the following.

THEOREM 1.4. *Let M be a compact connected 3-manifold, and let f be a weakly statically tame periodic homeomorphism $M \leftrightarrow M$. Then f is tame.*

There are well-known examples to show that a periodic homeomorphism of a compact 3-manifold need not be tame. See Montgomery and Zippin **[M-Z]** and Bing **[$\mathbf{B}_1$]**. In **[$\mathbf{B}_1$]** it is shown that f need not be tame even if F_1 is empty. In this example it is easy to see that F_{n_1} is wild for some n_1.

THEOREM 1.5. *Let f be a homeomorphism $\mathbf{S}^3 \leftrightarrow \mathbf{S}^3$, of period n, let F be the fixed-point set of f, and suppose that F is a tame 1-sphere. Then f is tame.*

If f is known by hypothesis to preserve orientation, then this follows immediately from Theorems 1.3 and 1.1 of **[M]**.

THEOREM 1.6. *Let f and F be as in Theorem 1.5, and suppose that F is unknotted. Then f is conjugate to a rotation.*

From this it follows that if f is a periodic homeomorphism $\mathbf{R}^3 \leftrightarrow \mathbf{R}^3$, with a straight line as its fixed-point set, then f is conjugate to a rotation. Thus Theorem 1.6 completes the solution of the problem of R. H. Bing that was cited in **[M]**. See **[$\mathbf{B}_3$**, p. 82**]**.

Theorems 1.5 and 1.6 will be proved in §7.

Special thanks are due to the referee, Professor Kyung W. Kwun, for discovering that my original argument for Theorem 1.4 was defective, as applied to the case in which M is not orientable and the period n is divisible by 4, and for sketching a remedy, whose crux is Theorem 6.2.

2. Locally Euclidean orbit spaces.

THEOREM 2.1. *Let N be a 3-manifold, and let g: $N \leftrightarrow N$ be a homeomorphism of period n, with fixed-point set G. Suppose that G is a compact tame 1-manifold, and that g has period exactly n at each point of $N - G$. Let Pr_g: $N \twoheadrightarrow \Omega_g$ be the projection of N onto the orbit space of g. Then Ω_g is a 3-manifold, and $\mathrm{Pr}_g\, G$ is tame in Ω_g.*

Here N is not required to be compact or connected, but G must be the union of a finite collection of 1-spheres J_i.

PROOF. Let $K(N)$ be a triangulation of N in which G forms a subcomplex, and in which the components J_i of G have disjoint regular neighborhoods N_i. Evidently each N_i is either a solid torus or a "solid Klein Bottle", according as N_i is or is not orientable. Since $g|(N - G)$ is an n-sheeted covering, it is obvious that every point of $\Omega_g - \mathrm{Pr}_g\, G$ has open Euclidean neighborhoods in Ω_g. Thus it will suffice to show that Ω_g is locally Euclidean at the points of $\mathrm{Pr}_g\, G$ and that $\mathrm{Pr}_g\, G$ is tame.

Case 1. Suppose that N_i is a solid torus. Let U_i be a connected open neighborhood of J_i and let $V_i = \bigcup_j g^j(U_i)$. We choose U_i sufficiently small so that $V_i \subset N_i$. Let $g_i = g|V_i$, let ϕ be a homeomorphism $N_i \to \mathbf{S}^3$, such that $\phi(J_i)$ is the boundary of a polyhedral 2-cell D in $\mathbf{S}^3$, let $M_i = \phi(V_i)$, and let f_i: $M_i \leftrightarrow M_i$ be defined by the condition $f_i(P) = \phi g_i \phi^{-1}(P)$. Then f_i is of period n, with fixed-point set $F_i = \phi(J_i)$, and f_i has period exactly n at each point of $M_i - F_i$. Thus M_i, f_i, F_i and n satisfy the hypothesis of Theorem 1.1 of [M]. Let Pr_i: $M_i \twoheadrightarrow \Omega_i$ be the projection of M_i onto the orbit space of f_i. By Theorem 1.1 of [M], Ω_i is a 3-manifold and $\mathrm{Pr}_i F_i$ is tame in Ω_i. But ϕ^{-1} induces a homeomorphism $\Omega_i \to \Omega_g$, of Ω_i onto an open set in Ω_g, with $\mathrm{Pr}_i \phi(P) \mapsto \mathrm{Pr}_g P$. Therefore Ω_g is locally Euclidean at each point of $\mathrm{Pr}_g\, J_i$, and $\mathrm{Pr}_g J_i$ is semilocally tame in Ω_g. (See §3 of [M].)

Case 2. Suppose that N_i is a solid Klein Bottle. As in Case 1, we take a connected open neighborhood U_i of J_i; we let $V_i = \bigcup_j g^j(U_i)$; let $g_i = g|V_i$; and choose U_i so that $V_i \subset N$. Let h: $\tilde{V}_i \twoheadrightarrow V_i$ be the 2-sheeted orientable covering of V_i. To get $\tilde{V}_i$ and h, we take a fixed point $P_0 \in J_i$, and take a fixed local orientation of V_i at P_0. Then the points of $\tilde{V}_i$ are equivalence classes $[p]$ of PL paths p: $[0, 1] \to V_i$, $0 \mapsto P_0$, $1 \mapsto P$. Two such paths are equivalent if they have the same terminal point P and induce the same local orientation at P. We then define $h([p]) = p(1)$. (For details of an almost identical construction, see [MGT, Theorem 21.7, p. 178].) If p_1 and p_2 are equivalent, then so also are $g(p_1)$ and $g(p_2)$. Therefore the homeomorphism g_i: $V_i \leftrightarrow V_i$ can be lifted so as to give a homeomorphism $\tilde{g}_i$: $\tilde{V}_i \leftrightarrow \tilde{V}_i$, with $\tilde{g}_i([p]) = [g_i(p)]$, so that $h(\tilde{g}_i) = g_i h$. Now the set $\tilde{J}_i = h^{-1}(J_i)$ is a 1-sphere in $\tilde{V}_i$, and is locally tame in $\tilde{V}_i$, because h is a local homeomorphism. Therefore $\tilde{J}_i$ is tame in $\tilde{V}_i$. $\tilde{J}_i$ is the fixed-point set of $\tilde{g}_i$, and $\tilde{g}_i$ has period exactly n at each point of $\tilde{V}_i - \tilde{J}_i$. Thus $\tilde{V}_i$, $\tilde{g}_i$, and $\tilde{J}_i$ are like the V_i, g_i, J_i of Case 1. Let $\Omega(\tilde{g}_i)$ be the orbit space of $\tilde{g}_i$, and let Pr'_i: $\tilde{V}_i \twoheadrightarrow \Omega(g_i)$ be the projection. By the result of Case 1, $\Omega(\tilde{g}_i)$ is a 3-manifold and $\mathrm{Pr}'_i \tilde{J}_i$ is tame in $\Omega(\tilde{g}_i)$. Now h: $\tilde{V}_i \to V_i$ induces a 2-sheeted covering h^*: $\Omega(\tilde{g}_i) \twoheadrightarrow \Omega_i$. To be precise, each point of $\Omega(\tilde{g}_i)$ is a finite set A of points of $\tilde{V}_i$, and $h^*(A) = \{h(P) | P \in A\}$; the latter set is a point of Ω_i. Thus Ω_i is a 3-manifold. Also, $h^*(\mathrm{Pr}'_i \tilde{J}_i) = \mathrm{Pr}_g J_i$ under the above definition, because the points of $h^*(\mathrm{Pr}'_i\tilde{J}_i)$ are the singletons $\{P\}$, where $P \in J_i$; these are the points of $\mathrm{Pr}_g J_i$. Since $\mathrm{Pr}'_i \tilde{J}_i$ is locally tame in $\Omega(\tilde{g}_i)$, it follows that $\mathrm{Pr}_g J_i$ is locally tame in Ω_i.

Thus Ω_g is a 3-manifold and $\mathrm{Pr}_g G = \bigcup_i \mathrm{Pr}_g J_i$ is locally tame in Ω_g. Therefore $\mathrm{Pr}_g G$ is tame in Ω_g, which was to be proved.

3. Powers of f, which have prime period. Let M, f, and n be as in Theorem 1.1. As in the discussion preceding Theorem 1.1, let $n = p_1 p_2 \cdots p_r$, where the p_i's are primes and $p_i \leqslant p_{i+1}$ for each $i < r$; for each i, let $n_i = n/p_i$ and let F_i be the fixed-point set of f^i. Then f^{n_i} has prime period p_i and we know by hypothesis that each set F_{n_i} is tame.

The following is the case $n = 3$ of a theorem of P. A. Smith [**S₁**]; see also A. Borel [**B₄**, Theorem 1, p. 76].

THEOREM 3.1 (P. A. SMITH). *Let N be a compact 3-manifold and let $g: N \leftrightarrow N$ be a homeomorphism of prime period p. Then the fixed-point set G of g is a manifold.*

This was stated for homological (or cohomological) manifolds in [**S₁**] and [**B₄**], but such spaces are locally Euclidean if their dimensions are 1 or 2. Here discrete sets and the empty set are being regarded as manifolds, of dimension 0 and -1 respectively.

THEOREM 3.2 (G. BREDON). *In Theorem 3.1, if p is odd and $G \neq \varnothing$, then G is a 1-manifold.*

This is a corollary of Theorem 2.3, on p. 77 of [**B₄**]. Thus F_{n_i} is always a manifold, and is a 1-manifold if p_i is odd.

THEOREM 3.3 (G. BREDON). *Let h be an involution of a compact connected 3-manifold onto itself, and let H be a component of the fixed-point set of h. Then H is of dimension 0 or 2 if and only if h reverses local orientation at some point of H.*

(This is a very special case of what Bredon proved. See [**B₃**, Theorem 2.5, p. 79].)

THEOREM 3.4. *Theorem 1.1 holds when n is an odd prime.*

PROOF. Let n be an odd prime, and let Pr be the projection of M onto the orbit space $\Omega(f)$. If the fixed-point set F_1 is $= \varnothing$, then Pr is a covering, and Theorem 1.1 follows. If $F_1 \neq \varnothing$, then it follows by Theorem 3.2 that F_1 is a 1-manifold. Since every point of $M - F_1$ has period exactly n, it follows that $\mathrm{Pr}|(M - F_1)$ is an n-fold covering, and that $\Omega(f)$ is locally Euclidean except perhaps at points of $\mathrm{Pr}\, F_1$. By Theorem 2.1, $\Omega(f)$ is locally Euclidean at each point of $\mathrm{Pr}\, F_1$, and $\mathrm{Pr}\, F_1$ is tame in $\Omega(f)$. Let K be a triangulation of $\Omega(f)$ in which $\mathrm{Pr}\, F_1$ forms a subcomplex, such that every simplex of K intersects $\mathrm{Pr}\, F_1$ in a simplex. Now K can be lifted so as to give a triangulation K' of M such that f is simplicial relative to K'.

Hereafter, it is to be understood that M, f, and F_i are as in Theorem 1.1.

THEOREM 3.5. *For each i, $f(F_i) = F_i$.*

Trivially: $P \in F_i \Leftrightarrow f^i(P) = P \Leftrightarrow f(f^i(P)) = f(P) \Leftrightarrow f^i(f(P)) = f(P) \Leftrightarrow f(P) \in F_i$.

THEOREM 3.6. *Let J_i and J_j be 1-spheres which are components of F_{n_i} and F_{n_j} respectively. If $J_i \cap J_j \neq \varnothing$, then $J_i = J_j$.*

PROOF. Suppose that $J_i \neq J_j$. Then there is an arc A in J_i such that $A \cap J_j$ is an end point P of A. Now $f^{n_j}(A)$ is an arc $A' \subset F_{n_i}$, with P as an end point. Since f^{n_j} is periodic and $A - P$ contains no point of F_{n_j}, it follows that $A' \cap A = P$. It then follows that $f^{n_j}(A') = A$ and $p_j = 2$. By logical symmetry, $p_i = 2$. Therefore $n_i = n_j$ and $F_{n_i} = F_{n_j}$, which contradicts the assumption $J_i \cap J_j \neq \varnothing$, $J_i \neq J_j$.

THEOREM 3.7. *Let J be a 1-sphere which is a component of F_{n_i}, and let T be a 2-manifold which is a component of F_{n_j}. If $P \in J \cap T$, then* (1) *J pierces T at P and* (2) *$J \cup T$ is tame.*

Evidently T separates every sufficiently small connected open neighborhood U of P. (1) means that if U is sufficiently small, then $U - T$ is the union of two disjoint open sets each of which intersects the component of $U \cap J$ that contains P.

LEMMA 1. *$J \cap T$ contains no arc.*

PROOF OF LEMMA. For convenience, we let $g = f^{n_i}$. Suppose that $J \cap T$ contains an arc A, and let $P_0 \in \operatorname{Int} A$. Let D_0 be a 2-cell neighborhood of P_0 in T such that $A \subset D_0$ and A decomposes D_0 into two 2-cells. We know that $g(T) = f^{n_i}(T) \subset F_{n_j}$, and since $g(P) = P$ it follows that $g(T) = T$. Let D_1 be another 2-cell neighborhood of P_0, intersecting A in an arc $A' = xy$, such that A' decomposes D_1 into two 2-cells, and sufficiently small so that $\bigcup_r g^r(D_1) \subset \operatorname{Int} D_0$.

Let $\tilde{V} = T - F_{n_i}$. Then $f^{n_i}|\tilde{V} = g|\tilde{V}$ has period exactly p_i at each point of $\tilde{V}$, and therefore is a covering. Let Pr be the projection of $\tilde{V}$ onto the orbit space V of $g|\tilde{V}$. Then V is a 2-manifold, and has a triangulation $K(V)$. Let $K(\tilde{V})$ be the lifting of $K(V)$ to $\tilde{V}$, so that $g|\tilde{V}$ is simplicial relative to $K(\tilde{V})$. Let L be the union of all simplices of $K(\tilde{V})$ that intersect $\bigcup_r g^r(D_1)$. Thus $L \cap A = \varnothing$. Since $\bigcup_r g^r(D_1)$ is g-invariant, so also is L. Let N be the regular neighborhood of L in $K(\tilde{V})$, that is, the star of L in the second barycentric subdivision of $K(\tilde{V})$. Then $g(N) = N$. We suppose that $K(V)$ was chosen at the outset in such a way that $N \subset \operatorname{Int} D_0$, and so that the diameters of the simplices of $K(\tilde{V})$ approach 0 as the simplices approach $T \cap F_{n_i}$. (These are conditions of "sufficient fineness".) From the latter condition it follows that $\bar{N} = N \cup A'$.

Since $D_1 - A$ has two components, lying on opposite sides of A in D_0, it follows that N has two components, N_1, N_2, lying on opposite sides of A in D_0. Each component of Bd N is a 1-sphere or is homeomorphic to a line. The 1-spheres can all be eliminated from Bd N; we merely add to N the 2-cells in D_0 which they bound. The resulting set is a g-invariant 2-manifold with boundary, lying in Int D_0, and $\bar{N} \cap A = A'$, as before. If B is a component of Bd N, then $\bar{B}$ has one of the forms $B \cup x$, $B \cup y$, $B \cup x \cup y$. If $\bar{B}$ is $B \cup x$ or $B \cup y$, then $\bar{B}$ bounds a 2-cell $D_B \subset D_0$, and Int $D_B \cap N = \varnothing$, because N_1 and N_2 are connected. We add such sets D_B to N. In the final N, every boundary component B is homeomorphic to a line and $\bar{B}$ is an arc between x and y. Since N_1 and N_2 are connected, each of them contains only one such B. Thus $\bar{N} = N \cup A'$ is a 2-cell and $g|\bar{N} = f^{n_i}|\bar{N}$ is a homeomorphism of period p_i, $\bar{N} \leftrightarrow \bar{N}$. Since $g|\bar{N}$ has an arc of fixed points, it must reverse orientation [**K**, p. 224]. Therefore $g|\bar{N}$ is of period 2 [**K**, p. 226].

But this is impossible. We have $p_i = 2$, and by Theorem 3.2, $p_j = 2$. Therefore $F_{n_i} = F_{n_j}$, which contradicts the hypothesis of the theorem. The lemma follows.

LEMMA 2. *$J - T$ has at most two components and $J \cap T$ contains at most two points.*

PROOF OF LEMMA. Let A be the closure of a component of $J - T$. If $A = J$, we are done. If not, A is an arc with its end points P, Q in T. Let $A' = f^{n_j}(A)$. As in the preceding proof, $A \cap A' = \{P, Q\}$, so that $A \cup A' = J$, and the lemma follows.

Since f^{n_j} is an involution, and T is tame, it follows that f^{n_j} is simplicial relative to some triangulation of a closed neighborhood of T. Thus, in the neighborhood of P, f^{n_j} looks like a reflection of 3-space across a plane. Since $f^{n_j}(J) = J$, (1) now follows. Similarly, J pierces T at the possible other points of $J \cap T$.

By hypothesis for F_{n_i}, F_{n_j}, the sets J and T are tame. It was shown in [$\mathbf{M}_7$] that if a tame arc e pierces a tame disk D at an interior point of D, then $D \cup e$ is tame. Thus P has a tame neighborhood in $J \cup T$. Similarly for any possible $Q \in J \cap T$. Therefore $J \cup T$ is locally tame. By the local tame imbedding theorem, (2) follows.

For later use, we note a stronger form of Theorem 1 of [$\mathbf{M}_7$].

THEOREM 3.8. *In a triangulated* 3-*manifold N, let D be a disk and let e be an arc, such that* (1) *$e \cap D$ is an interior point of D,* (2) *e pierces D at P,* (3) *e is tame, and* (4) *$D - P$ is locally tame. Then $D \cup e$ is tame.*

PROOF. We may assume that e is a linear interval, since e can be mapped onto an interval by a homeomorphism $N \leftrightarrow N$. Since $D - P$ is locally tame, it is tame. Let V be an open set containing $D - P$, such that $V \cap e = \varnothing$. By Theorem 3.4 of [**M**], there is a homeomorphism h: $N \leftrightarrow N$ such that $h(D - P)$ is a polyhedron and $h|(N - V)$ is the identity. Theorem 2 of [$\mathbf{M}_7$] asserts that under these conditions $h(D \cup e) = h(D) \cup e$ is tame. Therefore so also is $D \cup e$.

THEOREM 3.9. *Let T_i and T_j be* 2-*manifolds which are components of F_{n_i} and F_{n_j} respectively. If $T_i \cap T_j \neq \varnothing$, then $T_i = T_j$.*

PROOF. Here f^{n_i} and f^{n_j} must both be of period 2. Therefore $p_i = p_j$, $n_i = n_j$, and $F_{n_i} = F_{n_j}$.

Since we know that each set F_{n_i} is tame, and every locally tame set is tame, Theorem 3.8 gives the following.

THEOREM 3.10. *The set $\bigcup_i F_{n_i}$ is tame. In the space $\bigcup_i F_{n_i}$, every point which is not isolated has an open neighborhood of one of the following types*: (1) *the interior of a disk,* (2) *the interior of an arc or* (3) Int $D \cup$ Int e, *where e is an arc piercing D at an interior point P. In* (3), *D and e lie in different sets F_{n_i}, F_{n_j}.*

A set which satisfies the conditions of the conclusion of Theorem 3.10 will be called a *mixed manifold with piercing singularities* (MMPS). The word *mixed* refers to the fact that such a set may be the union of a collection of manifolds of different dimensions ranging from 0 to 2.

The following must be known, but I know of no reference.

THEOREM 3.11. *Let T be a connected 2-manifold, let g be a homeomorphism $T \leftrightarrow T$ of prime period p, and let P be an isolated fixed point of g. Let $\Omega = \Omega(g)$ be the orbit space of g and let* Pr: $T \to \Omega$ *be the projection. Then there is a 2-cell neighborhood D, of P in N, such that* (1) $g(D) = D$ *and* (2) Pr D *is a 2-cell neighborhood of* Pr P *in* Ω.

PROOF. Let W be an open neighborhood of P in T and let $V = \bigcup_i g^i(W)$, so that $g(V) = V$. Then $\mathrm{Pr}|(V - P)$ is a p-sheeted covering, so that $\mathrm{Pr}(V - P)$ is a 2-manifold. Also Pr V is open in Ω. We choose W sufficiently small so that V lies in a Euclidean neighborhood of P in T. It has been shown by G. T. Whyburn that the image of a 2-manifold under a light interior mapping is a 2-manifold, possibly with boundary. (See G. T. Whyburn [W, Theorem 4.4, p. 197]. I am indebted to the referee for the reference.) Since Bd Pr V has no isolated points, and Bd $\mathrm{Pr}(V - P) = \varnothing$, it follows that Bd Pr $V = \varnothing$. Let D' be a 2-cell which is a neighborhood of Pr P in Ω and let $D = \mathrm{Pr}^{-1}D'$.

4. The partial orbit space sequence $\Omega_1, \Omega_2, \ldots, \Omega_{r-q}$. We recall that $n = p_1 p_2 \cdots p_r$, where $p_{i+1} \geqslant p_i$; $n_i = n/p_i$; and F_{n_i} is the fixed-point set of f^{n_i}: $M \leftrightarrow M$. For $1 \leqslant i \leqslant r$, let

$$m_i = n/p_{r-i+1}p_{r-i+2} \cdots p_r,$$

so that $m_1 = n/p_r$, $m_r = 1$, and $m_i = p_1 p_2 \cdots p_{r-i}$ for $1 < i < r$. Let Pr_i: $M \to \Omega_i$ be the projection of M onto the orbit space of f^{m_i}. For each finite set A, let $|A|$ be the number of elements in A. As usual, for integers a, b, $a|b$ means that a divides b. Now the points of Ω_i are sets of the type

$$\mathrm{Pr}_i\, P = \{f^{jm_i}(P) | P \in M, j \in \mathbf{Z}\}.$$

Since f^{m_i} has period $n/m_i = p_{r-i+1}p_{r-i+2} \cdots p_r$, it follows that for each i and each P, $|\mathrm{Pr}_i\, P|\ |(n/m_i)$; the reason is that $|\mathrm{Pr}_i\, P|$ is the period of f^{m_i} at P. It is clear that

$$f(\mathrm{Pr}_i\, P) = f(\{f^{jm_i}(P)\}) = \{f^{jm_i}(f(P))\} = \mathrm{Pr}_i\, f(P).$$

Thus f induces a homeomorphism f_*: $\Omega_i \leftrightarrow \Omega_i$. Consider now $f_*^{m_{i+1}}$: $\Omega_i \leftrightarrow \Omega_i$. This is a homeomorphism of period p_{r-i}. Let Ω_{i+1}^* be the orbit space of $f_*^{m_{i+1}}$ and let Pr_{i+1}^* be the projection $\Omega_i \to \Omega_{i+1}^*$. Thus a point of Ω_{i+1}^* is a collection of sets of the form

$$\mathrm{Pr}_{i+1}^* \mathrm{Pr}_i\, P = \mathrm{Pr}_{i+1}^*\{f^{jm_i}(P)\} = \{f^{km_{i+1}}(\{f^{jm_i}(P)\})\},$$

where j and k are any integers. Consider now the union W of the elements of the above collection. All the elements Q of W are points of the type $f^s(P)$: if $m_{i+1}|s$, then $Q \in W$; and the converse also holds, because $f^{km_{i+1}}(f^{jm_i}(P)) = f^{km_{i+1}+jm_i}(P)$ and $m_{i+1}|m_i$. Thus each W is simply a point of Ω_{i+1}. We define

$$h_i \mathrm{Pr}_{i+1}^*\, \mathrm{Pr}_i\, P = W = \{f^{jm_{i+1}}(P)\} \in \Omega_{i+1}.$$

Thus we have a homeomorphism h_i: $\Omega_{i+1}^* \leftrightarrow \Omega_{i+1}$, and for practical purposes Ω_i^* and Ω_i can be regarded as indistinguishable.

For each i, let G_i be the fixed-point set of $f_*^{m_{i+1}}: \Omega_i \leftrightarrow \Omega_i$. For convenience, we define Ω_0 to be M and we let Pr_0 be the identity $\Omega_0 \leftrightarrow \Omega_0$, $P \mapsto P$. Thus G_0 is the fixed-point set of $f^{m_{0+1}} = f^{m_1}: \Omega_0 \leftrightarrow \Omega_0$.

THEOREM 4.1. *For each i, $G_i \subset \mathrm{Pr}_i\, F_{n_{r-i}}$.*

PROOF. Let $\mathrm{Pr}_i\, P = \{f^{jm_i}(P)\}$ and let $k = |\mathrm{Pr}_i\, P|$. Then $k|n/m_i$. If $\mathrm{Pr}_i\, P \in G_i$, then $|\mathrm{Pr}_{i+1}\, P| = k$. Now f_* is a homeomorphism $\Omega_{i+1} \leftrightarrow \Omega_{i+1}$, of period $m_{i+1} = p_1 p_2 \cdots p_{r-i-1}$. Therefore $|\{f_*^j(\mathrm{Pr}_{i+1}\, P)\}|\, |m_{i+1}$. But $|\mathrm{Pr}_r(P)|\, |km_{i+1}$, and $km_{i+1}|(n/m_i)m_{i+1} = n/p_{r-i} = n_{r-i}$. Therefore $|\mathrm{Pr}_r\, P|\, |n_{r-i}, f^{n_{r-i}}(P) = P$, $P \in F_{n_{r-i}}$, and $\mathrm{Pr}_i\, P \in \mathrm{Pr}_i\, F_{n_{r-i}}$. Thus $G_i \subset \mathrm{Pr}_i\, F_{n_{r-i}}$, which was to be proved.

THEOREM 4.2. *For each i, s, and t,*

$$(\mathrm{Pr}_i\, F_{n_s}) \cap (\mathrm{Pr}_i\, F_{n_t}) = \mathrm{Pr}_i(F_{n_s} \cap F_{n_t}).$$

PROOF. This holds for $i = 0$. We shall show that if i has the stated property, then so also does $i + 1$. Since the sets F_{n_k} are f-invariant, so also is each set $\mathrm{Pr}_i\, F_{n_k}$ and so also is $\bigcup_k \mathrm{Pr}_i\, F_{n_k}$. Now consider $f_*^{m_{i+1}}: \Omega_i \leftrightarrow \Omega_i$, with fixed-point set $G_i \subset \mathrm{Pr}_i\, F_{n_{r-i}}$. Evidently $\mathrm{Pr}_{i+1}^*|(\Omega_i - G_i)$ is a covering, and thus is a local homeomorphism. Since $\bigcup_k \mathrm{Pr}_i\, F_{n_k}$ is f-invariant, and so also is G_i, it follows that if the theorem fails, it fails at some point of $\mathrm{Pr}_{i+1}\, G_i$. But if $P \in (\mathrm{Pr}_{i+1}\, F_{r_s}) \cap (\mathrm{Pr}_{i+1}\, F_{n_t}) \cap \mathrm{Pr}_i\, G_i$, then $P = \mathrm{Pr}_i\, Q$ for some $Q \in (\mathrm{Pr}_i\, F_{n_s}) \cap (\mathrm{Pr}_i\, F_{n_t})$. By the induction hypothesis, the theorem follows.

So far, we have not used or needed the hypothesis that $p_i < p_{i+1}$ for $i < r$, but we shall need it from now on. Let q be the largest integer such that $2^q|n$. Thus $n = 2^q p_{q+1} p_{q+2} \cdots p_r$.

THEOREM 4.3. *If $0 \leqslant i \leqslant r - q$, then*:

(1) *Ω_i is a 3-manifold.*

(2) *For each s, $\mathrm{Pr}_i\, F_{n_s}$ is a tame manifold in Ω_i.*

(3) *Each component C of $\mathrm{Pr}_i\, F_{n_s}$ is of the form $\mathrm{Pr}_i\, C'$, where C' is a component of F_{n_s}.*

(4) *If C_s and C_t are components of $\mathrm{Pr}_i\, F_{n_s}$ and $\mathrm{Pr}_i\, F_{n_t}$ respectively, then C_s' and C_t' can be chosen so that $C_s \cap C_t = \mathrm{Pr}_i(C_s' \cap C_t')$.*

(5) *$\bigcup_s \mathrm{Pr}_i\, F_{n_s}$ is tame in Ω_i.*

(6) *If J is a 1-dimensional component of $\mathrm{Pr}_i\, F_{n_s}$, and T is a 2-dimensional component of $\mathrm{Pr}_i\, F_{n_t}$, and $P \in J \cap T$, then J pierces T at P.*

PROOF. For $i = 0$, all this is known. Inductively, we shall show that if i satisfies (1)–(6) and $i < r - q$, then $i + 1$ satisfies (1)–(6). The length of this theorem may seem awkward, but all six of the conclusions will be needed as induction hypotheses.

Suppose that Ω_i and Pr_i satisfy (1)–(6), with $i < r - q$. Now $f_*^{m_{i+1}}$ is a homeomorphism $\Omega_i \leftrightarrow \Omega_i$, of odd prime period p_{r-i}, with fixed-point set G_i. If $G_i = \varnothing$, then $f^{m_{i+1}}$ is a covering. Since all the sets F_{n_s} are f-invariant, it is a routine matter to verify that (1)–(6) are preserved when we pass from i to $i + 1$ ($i + 1 \leqslant r - q$). Hereafter we assume that $G_i \neq \varnothing$.

Suppose, then, that $G_i \neq \varnothing$. By Theorem 3.2 it follows that G_i is a 1-manifold. Since $f^{n_{r-i}}$ has odd prime period p_{r-i}, its fixed-point set $F_{n_{r-i}}$ is a 1-manifold; and by our induction hypotheses, $\mathrm{Pr}_i\, F_{n_{r-i}}$ is tame. By Theorem 4.1, $G_i \subset \mathrm{Pr}_i\, F_{n_{r-i}}$, and so G_i is the union of a (finite) collection of components of $\mathrm{Pr}_i\, F_{n_{r-i}}$. Therefore G_i is tame in Ω_i. By Theorem 2.1, $\mathrm{Pr}^*_{i+1}\, G_i$ is tame in Ω^*_{i+1}. Since h_i: $\Omega^*_{i+1} \leftrightarrow \Omega_{i+1}$ is a homeomorphism and $h_i \mathrm{Pr}^*_{i+1} = \mathrm{Pr}_{i+1}$, it follows that (1) is satisfied by $i + 1$ and that $\mathrm{Pr}_{i+1}\mathrm{Pr}_i^{-1}\, G_i$ is tame in Ω_{i+1}.

Consider the set $\bigcup_s \mathrm{Pr}_i\, F_{n_s}$. Since each set F_{n_s} is f-invariant, it follows that each set $\mathrm{Pr}_i\, F_{n_s}$ is $f_*^{m_{i+1}}$-invariant. Let J_s and J_t be 1-spheres which are components of $\mathrm{Pr}_i\, F_{n_s}$ and $\mathrm{Pr}_i\, F_{n_t}$ respectively. If $J_s \cap J_t \neq \varnothing$, then by (3) and (4) it follows that $J_s = \mathrm{Pr}_i\, J'_s$, $J_t = \mathrm{Pr}_i\, J'_t$, where J'_s and J'_t are components of F_{n_s} and F_{n_t} and $J'_s \cap J'_t \neq \varnothing$. By Theorem 3.6 it follows that $J'_s = J'_t$. Therefore $J_s = J_t$. Thus *the union of the* 1-*dimensional components of* $\bigcup_s \mathrm{Pr}_i\, F_{n_s}$ *is a* 1-*manifold.* Since these components J are disjoint and their union is $f_*^{m_{i+1}}$-invariant, it follows that $f_*^{m_{i+1}}$ merely permutes them (leaving components of G_i fixed, of course). By the induction hypothesis, each of these components J is tame and therefore locally tame. If $J \subset G_i$, we already know that $h_i \mathrm{Pr}^*_{i+1}\, J$ is tame in Ω_{i+1}. If $J \cap G_i = \varnothing$, then it follows by much the same reasoning that $h_i \mathrm{Pr}^*_{i+1}\, J = \mathrm{Pr}_{i+1}\mathrm{Pr}_i^{-1}\, J$ is locally tame and hence tame in Ω_{i+1}. Thus we have a restricted form of (2): every 1-dimensional component of $\mathrm{Pr}_{i+1}\, F_{n_s}$ is tame in Ω_{i+1}. Note also that (3), (4) and (5) are satisfied by $i + 1$ if only 1-dimensional components are considered.

To complete the proof of (2), it remains to consider the 2-dimensional components of $\bigcup_s \mathrm{Pr}_{i+1}\, F_{n_s}$. Let W_i (and W_{i+1}) be the union of the 2-dimensional components of $\bigcup_s \mathrm{Pr}_i\, F_{n_s}$ (and $\bigcup_s \mathrm{Pr}_{i+1}\, F_{n_s}$). Each component T of W_i is a set of the type $\mathrm{Pr}_i\, T'$, where T' is a component of a set F_{n_s} (condition (3) of the theorem). Since F_{n_s} is f-invariant, it follows as before that the manifold $\mathrm{Pr}_i\, F_{n_s}$ is $f_*^{m_{i+1}}$-invariant. Thus $\bigcup_j f_*^{jm_{i+1}}(T)$ is a 2-manifold. If $T \cap G_i = \varnothing$, then it follows that $f_*^{m_{i+1}}$ merely permutes the components of $\bigcup_j f_*^{jm_{i+1}}(T)$. It follows, as for the 1-spheres above, that $h_i(\mathrm{Pr}^*_{i+1}\, T)$ is a component of $\mathrm{Pr}_{i+1}\, F_{n_s}$; this set is a 2-manifold, and is locally tame in Ω_{i+1} and therefore tame.

Thus the case of interest is the one in which T intersects G_i at one or more points P. Let J be the component of G_i that contains P. We already know that $h_i(\mathrm{Pr}^*_{i+1} J)$ is tame in Ω_{i+1}, and it is easy to check (as in earlier cases) that $h_i(\mathrm{Pr}^*_{i+1}(T - G_i))$ is locally tame. We shall show that $\mathrm{Pr}^*_{i+1}\, P$ has a neighborhood in $\mathrm{Pr}^*_{i+1}(T \cup J)$ which is tame in Ω^*_{i+1}.

By condition (6) for i, J pierces T at P. By Theorem 3.11, there is a 2-cell neighborhood D of P in T such that $f_*^{m_{i+1}}(D) = D$ and such that $\mathrm{Pr}^*_{i+1}\, D$ is a 2-cell neighborhood of $\mathrm{Pr}^*_{i+1}\, P$ in $\mathrm{Pr}^*_{i+1}\, T$. Since $T \cup J$ is tame (condition (5) for i), it follows that P has a 3-cell neighborhood C^3 in Ω_i such that D decomposes C^3 into two 3-cells C_1^3, C_2^3, which intersect J in arcs A_1, A_2, with $A_1 \cap A_2 = P$. For $k = 1, 2$, let $U_k = C_k^3 \cap \mathrm{Int}\, C_3$. Then $\bigcup_j f_*^{jm_{i+1}}(U_1 \cup U_2) = \bigcup_j f_*^{jm_{i+1}}(\mathrm{Int}\, C^3)$ and this set, call it V, is a connected neighborhood of P in Ω_i. Obviously V is the union of the sets $V_k = \bigcup_j f_*^{jm_{i+1}}(U_k)$ $(k = 1, 2)$, and $V_1 \cap V_2 = \mathrm{Int}\, D$. Now $\mathrm{Pr}^*_{i+1}\, V$ is a connected open set in Ω^*_{i+1}, containing $\mathrm{Pr}^*_{i+1}\, P$, and $\mathrm{Pr}^*_{i+1}\, V$ is decomposed by

$\mathrm{Pr}^*_{i+1} D$ into two connected open sets, each of which intersects $\mathrm{Pr}^*_{i+1} J$. It follows that $\mathrm{Pr}^*_{i+1} J$ pierces $\mathrm{Pr}^*_{i+1} D$ at $\mathrm{Pr}^*_{i+1} P$. Since $\mathrm{Pr}^*_{i+1}(D - P)$ is locally tame, it is tame. By Theorem 3.8, $\mathrm{Pr}^*_{i+1}(D \cup A_1 \cup A_2)$ is tame.

We have considered various types of components of sets $\mathrm{Pr}_i F_{n_s}$; their images under $h_i(\mathrm{Pr}^*_{i+1})$, in each case, have been components of $\mathrm{Pr}_{i+1} F_{n_s}$. Thus (3) always holds for the integer $i + 1$. Similarly for (4). (2) has been verified for all components of all sets $\mathrm{Pr}_{i+1} F_{n_s}$. We have shown that each set $\mathrm{Pr}_{i+1} F_{n_s}$ is tame; and in the preceding paragraph we completed the proof that (5) and (6) hold for $i + 1$. This completes the proof of the theorem.

Setting $i = r - q$ in Theorem 4.3, we get the following.

THEOREM 4.4. *Ω_{r-q} has a triangulation $K(\Omega_{r-q})$ such that* (1) *each set* $\mathrm{Pr}_{r-q} F_{n_s}$ *forms a subcomplex of $K(\Omega_{r-q})$ and* (2) *each simplex of $K(\Omega_{r-q})$ intersects* $\bigcup_s \mathrm{Pr}_{r-q} F_{n_s}$ *in a simplex.*

(To get a complex satisfying (2), we may need to subdivide a complex satisfying (1).)

Note that for the case $q = 0$, we have $\Omega_{r-q} = \Omega_r$ and the proof of Theorem 1.4 is already complete. We lift $K(\Omega_r)$ in the obvious way to get a triangulation $K(M)$ relative to which f is simplicial.

Hereafter we suppose that $q > 0$; and we need to show that Ω_r has a triangulation $K(\Omega_r)$ as in Theorem 4.4. The proofs of Theorems 4.3 and 4.4 use the local tame imbedding theorem, which applies, as stated, only in 3-manifolds; and there are simple examples to show that Ω_r need not be a manifold or even a manifold with boundary. Hence the digression in the following section.

5. Local properties of involutions. Throughout this section, M^3 will be a compact connected 3-manifold, and $h\colon M^3 \leftrightarrow M^3$ will be an involution. For $i = 0, 1, 2$, M^i will be either the empty set or a tame i-manifold in M^3, compact but not necessarily connected. (Here by a 0-manifold we mean a discrete set.) We assume that $h(M^i) = M^i$ for each i and that $M^0 \cup M^1 \cup M^2$ contains the fixed-point set F_h of h. We assume also that each component J of M^1 pierces M^2 at each point P of $J \cap M^2$. Pr_h will be the projection of M^3 onto the orbit space Ω_h of h.

We shall investigate the action of h in small neighborhoods of fixed points. The results will be applied in the following section to the case in which (1) $M^3 = \Omega_{r-i}$, (2) $M^0 \cup M^1 \cup M^2 = \mathrm{Pr}_{r-i} \bigcup_j F_{n_j}$ and (3) $h = f_*^{m_{i+1}}\colon \Omega_{r-i} \leftrightarrow \Omega_{r-i}$, where $i > r - q$. But in this section, the notation and apparatus of §4 would be a needless burden.

For each set S of points and each point $v \notin S$, Sv denotes the join of S and v with the usual topology. By σ^n we shall always mean an n-simplex. Thus $\sigma^2 v$ is always a 3-cell, and if P is a point, then Pv is an arc. (We shall still use the notation PQ for any arc from P to Q.)

By a *star-complex* we mean a complex K which is the star of one of its own vertices. If K is a star-complex, then K will be called a *star-triangulation* of the corresponding polyhedron $|K|$. If $K = \mathrm{St}\, P$, then P will be called a *central vertex* of K.

THEOREM 5.1. *Let $\tilde{U} = M^3 - F_h$. Then $\tilde{U}$ has a triangulation $K(\tilde{U})$ such that* (1) *$h|\tilde{U}$ is simplicial relative to $K(\tilde{U})$ and* (2) *$(M^1 \cup M^2) \cap \tilde{U}$ is a polyhedron relative to $K(\tilde{U})$.*

PROOF. Since M^1 and M^2 are tame and M^1 pierces M^2 at every intersection point, it follows by Theorem 3.8, together with the local tame imbedding theorem, that the set $W = M^1 \cup M^2$ is tame. Let

$$\rho = \mathrm{Pr}_h|\tilde{U}: \tilde{U} \twoheadrightarrow U = \mathrm{Pr}_h\, \tilde{U}.$$

Then ρ is a covering, and hence is a local homeomorphism, and U is a 3-manifold. Since $h(W \cap \tilde{U}) = W \cap \tilde{U}$, it follows that $\rho|W \cap \tilde{U}$ is a local homeomorphism. Therefore $\rho(W \cap \tilde{U})$ is locally tame, and hence tame, in U. Let $K(U)$ be a triangulation of U and let ψ: $U \leftrightarrow U$ be a homeomorphism such that $\psi\rho(W \cap \tilde{U})$ is a polyhedron relative to $K(U)$. Let

$$K'(U) = \psi^{-1}(K(U)) = \{\psi^{-1}(\sigma)|\sigma \in K(U)\}.$$

Then $\rho(W \cap \tilde{U})$ is a polyhedron relative to $K'(U)$. Let $K(\tilde{U})$ be the lifting of $K'(U)$ to $\tilde{U}$. Then $K(\tilde{U})$ satisfies both (1) and (2).

THEOREM 5.2. *In a triangulated* 3-*manifold M^3, let $B_1, B_2, \ldots, B_r, A$ be annuli, each two of which intersect in a* 1-*sphere J which is a component of the boundary of each. If each B_i is a polyhedron and $A - J$ is polyhedral, then $A \cup \bigcup_i B_i$ is tame. In fact, for each open set V containing J there is a homeomorphism g: $M^3 \leftrightarrow M^3$ such that $g(A)$ is a polyhedron and g is the identity on each B_i and on $M^3 - V$.*

PROOF. For the case in which M^3 is 3-space and $r = 1$, this is Lemma 5.2, on p. 165 of **[M8]**. The proof was by construction of a homeomorphism which differed from the identity only in a small neighborhood of J. The same proof therefore works in an arbitrary M^3. In the proof, we show that we can "straighten out A", so as to get a polyhedron, without needing to move any point of another polyhedral annulus B_1 as in Theorem 5.2. By the same procedure, we can leave all the annuli B_i pointwise fixed. See Lemma 5.1, p. 164 of **[M$_8$]**, which generalizes Lemma 5.2 in a similar way.

THEOREM 5.3. *In a triangulated* 3-*manifold M^3, let $A_1, A_2, \ldots, A_r$ be annuli, and let J be a* 1-*sphere which is a component of the boundary of each of them and is the intersection of every two of them. If J is tame and each set $A_i - J$ is tame, then $\bigcup_i A_i$ is tame.*

PROOF. Let g_1 be a homeomorphism $M^3 \leftrightarrow M^3$ such that $g_1(J)$ is a polyhedron. Let g_2 be a homeomorphism $M^3 \leftrightarrow M^3$ such that $g_2|J$ is the identity and $g_2g_1(A_1 - J)$ is a polyhedron. Lemma 2.1 on pp. 159–160 of **[M$_8$]** asserts that under these conditions there is a homeomorphism g_3: $M^3 \leftrightarrow M^3$ such that $g_3g_2g_1(A_1)$ is a polyhedron. (This lemma was stated only for 3-space, but the generalization is immediate, just as for Lemma 5.2.) Now let g_4 be a homeomorphism $M^3 \leftrightarrow M^3$ such that $g_4g_3g_2g_1(A_2 - J)$ is a polyhedron and $g_4|g_3g_2g_1(A_1)$ is the identity. By Theorem 5.2 we can move $g_4g_3g_2g_1(A_2)$ onto a polyhedron, leaving $g_4g_3g_2g_1(A_1)$ pointwise fixed.

In a finite number of such steps, we move $\bigcup_i A_i$ onto a polyhedron. Thus $\bigcup_i A_i$ is tame.

We return to the discussion of the M^3 and h described at the beginning of this section. F_h and the sets M^i are as before.

THEOREM 5.4. *Let H be a component of M^2, lying in F_h, and let J be a component of M^1, piercing H at a point P. Then P has a 3-cell neighborhood C in M^3 with the following properties.*

(1) $h(C) = C$.

(2) Bd $C \cup M^1 \cup M^2$ *is tame in* M^3.

(3) M^1 *pierces* Bd C *at exactly two points, and intersects* Bd C *only at these points.*

(4) $C \cap M^2$ *is a 2-cell D such that* $D \cap$ Bd $C =$ Bd D.

(5) $\text{Pr}_h\, C$ *has a star-triangulation in which* $\text{Pr}_h\, D$ *and* $\text{Pr}_h(M^1 \cap C)$ *form subcomplexes and in which* $\text{Pr}_h\, P$ *is a central vertex.*

PROOF. Since H is a 2-manifold in F_h, it follows that h reverses local orientation at P. Let D be a "small" h-invariant 2-cell neighborhood of P in M^2. Since M^2 is tame, D is automatically tame. Let ϕ be a homeomorphism $Dv \leftrightarrow C_1 \subset M^3$, so that C_1 is a 3-cell. We choose ϕ in such a way that (a) C_1 is tame, (b) $C_1 \cap M^2 = D \subset$ Bd C_1, (c) $C_1 \cap M^1$ is an arc PQ such that M^1 pierces Bd C_1 at P and Q, and (d) $PQ = \phi(Pv)$. Under these conditions, C_1 has a star-triangulation K_1 in which PQ is an edge and D forms a subcomplex; in fact, K_1 is the star of P in K_1. Let $C = C_1 \cup h(C_1)$. Then $\text{Pr}_h|C_1$ is a homeomorphism $C_1 \leftrightarrow \text{Pr}_h\, C$, so that $\text{Pr}_h\, K_1 = \{\text{Pr}_h\, \sigma | \sigma \in K_1\}$ is the star-triangulation desired in (5). It is clear by construction that (1), (3), and (4) hold. To verify (2), we note that since Bd C_1, Bd $h(C_1)$, and $M^1 \cup M^2$ are all tame, it follows that Bd $C \cup M^1 \cup M^2$ is locally tame except perhaps at points of J and at points of Bd $C \cap M^1$. At points of J, use Theorem 5.3. At points of Bd $C \cap M^1$, use Theorem 3.8. Finally, use the local tame imbedding theorem.

THEOREM 5.5. *Let $P \in M^2 - M^1$, and suppose that P is an isolated point of F_h. Then P has arbitrarily small 3-cell neighborhoods C in M^3 with the following properties.*

(1) $h(C) = C$.

(2) $C \cup M^1 \cup M^2$ *is tame in* M^3.

(3) $C \cap M^2$ *is a 2-cell D such that* $D \cap$ Bd $C =$ Bd D.

(4) $\text{Pr}_h\, C$ *has a star-triangulation in which* $\text{Pr}_h\, P$ *is a central vertex and* $\text{Pr}_h\, D$ *forms a subcomplex.*

PROOF. Let E be a "small" 2-cell neighborhood of P in M^2. Let $\tilde{U} = M^3 - F_h$ and let $K(\tilde{U})$ be as in Theorem 5.1. Let N be a connected closed neighborhood of P in M^2, such that the frontier of N in the space M^2 is a polyhedron relative to $K(\tilde{U})$, and sufficiently small so that

$$N' = N \cup h(N) = \text{Pr}_h^{-1}\text{Pr}_h\, N \subset \text{Int}\, E.$$

Now $h(N') = N'$, and the frontier of N' in M^2 is polyhedral relative to $K(\tilde{U})$. (Recall from Theorem 5.1 that $h|\tilde{U}$ is simplicial relative to $K(\tilde{U})$.) Therefore this frontier forms a subcomplex of a subdivision $K'(\tilde{U})$ of $K(\tilde{U})$ relative to which $h|\tilde{U}$ is simplicial. If N'' is a regular neighborhood of $N' - P$ in an nth barycentric subdivision $b^nK'(U)$, then we have $h(N'') = N''$; and if N'' is a sufficiently small neighborhood of $N' - P$, then $N'' \subset \text{Int } E$. We choose an N'' satisfying both these conditions. Now let D be the union of N'', $\{P\}$, and all components of $E - N''$ except the one that contains Bd E. Then D is a 2-cell, $h(D) = D$, and $D - P$ is polyhedral relative to $K(\tilde{U})$.

Now h reverses local orientation of M^3 at P; since $D \cap F_h = P$, $h|D$ reverses orientation in D. Let C_1 be a 3-cell in M^3 such that

$$C_1 \cap M^2 = \text{Bd } C_1 \cap M^2 = D$$

and such that Bd $C_1 - P$ is polyhedral relative to $K(\tilde{U})$. Let $C = C_1 \cup h(C_1)$. If C_1 lies in a sufficiently small neighborhood (and $D \cap M^1 = \varnothing$), then C will be a 3-cell, with $C \cap M^1 = \varnothing$. Since h is simplicial relative to $K(\tilde{U})$, Bd $h(C_1) - P$ is a polyhedron. Therefore so also is Bd C.

Now $h|D$ is conjugate to an antipodal mapping $\mathbf{B}^2 \leftrightarrow \mathbf{B}^2$, where $\mathbf{B}^2$ is the unit disk in $\mathbf{R}^2$. Therefore there is a triangulation $K(D)$ of D in which D forms the (closed) star of P, such that $h|D$ is simplicial relative to $K(D)$. Let $K(C_1)$ be a triangulation of C_1 which forms the join of $K(D)$ with a point $v \in \text{Bd } C_1 - D$; and let $K(C) = K(C_1) \cup h(K(C_1))$. (Here by $h(K(C_1))$ we mean $\{h(\sigma)|\sigma \in K(C_1)\}$.) Let $K(\text{Pr}_h\ C) = \{\text{Pr}_h\ \sigma|\sigma \in K(C)\}$. Then C and $K(\text{Pr}_h\ C)$ are as desired in Theorem 5.2. (The verification of (2) is based on Theorem 5.3, as in the proof of Theorem 5.4.)

THEOREM 5.6. *Let $P \in M^1 \cap M^2$, and suppose that P is an isolated point of F_h. Then P has arbitrarily small* 3*-cell neighborhoods C, satisfying conditions* (1)–(4) *of Theorem* 5.5, *such that also*

(5) *$M^1 \cap C$ forms a subcomplex of the star-triangulation mentioned in* (4) *and*

(6) *M^1 pierces* Bd *C at exactly two points and intersects* Bd *C only at these points.*

The proof is the same as that of Theorem 5.5, except that in defining C_1 we need to take note of M^1 and take the obvious precautions, as in the proof of Theorem 5.4.

THEOREM 5.7. *Let $P \in M^1 - M^2$, and suppose that P is an isolated point of F_h. Then P has arbitrarily small* 3*-cell neighborhoods C satisfying the following conditions.*

(1) $h(C) = C$.

(2) *$C \cup M^1 \cup M^2$ is tame in M^3.*

(3) *M^1 pierces* Bd *C at exactly two points, and intersects* Bd *C only at these points.*

(4) Pr_h *C has a star-triangulation in which* $\text{Pr}_h(C \cap M^1)$ *is an edge and* Pr_h *P is a central vertex.*

PROOF. First we consider a related situation in Cartesian 3-space $\mathbf{R}^3$. Let V be an open set in $\mathbf{R}^3$, with frontier Fr V. Let I be an open linear interval in V, with its end points in Fr V, let $Q \in I$, and let g be an involution $V \leftrightarrow V$ with Q as its only fixed point, such that $g(I) = I$. Let Pr_g be the projection of V onto the orbit space

Ω_g, let $\tilde{W} = V - I$, and let $W = \Pr_g \tilde{W}$. Let $K(W)$ be a triangulation of W such that the diameters of the simplices of $K(W)$ approach 0 as the simplices approach $\Pr_g I$, and let $K(\tilde{W})$ be the lifting of $K(W)$ to $\tilde{W}$.

LEMMA 5.7.1. *Under the above conditions, Q has arbitrarily small 3-cell neighborhoods C_1 such that*

(1) $g(C_1) = C_1$.

(2) *I pierces* Bd C_1 *at exactly two points and intersects* Bd C_1 *only at these points.*

(3) Bd $C_1 - I$ *is polyhedral relative to* $K(\tilde{W})$.

(4) Bd $C_1 \cup I$ *is tame in* V.

PROOF OF LEMMA. By Theorem 3.8, (4) is a consequence of (2) and (3). We shall show that there are arbitrarily small 3-cells satisfying (1), (2), and (3).

Let $R \in I - Q$ and let $R' = g(R)$. Then every neighborhood of R contains a 2-cell $\tilde{D}$ such that (a) I pierces $\tilde{D}$ at R and (b) $\tilde{D} - R$ is a polyhedron relative to $K(\tilde{W})$. We choose $\tilde{D}$ in a sufficiently small neighborhood of R so that $\Pr_g|\tilde{D}$ is a homeomorphism and $g(\tilde{D}) \cap \tilde{D} = \varnothing$. Let $D = \Pr_g \tilde{D}$. Thus $\Pr_g^{-1} D = \tilde{D} \cup \tilde{D}'$, where $\tilde{D}'$ is a 2-cell which is pierced by I at R'.

Let RR' be the closed interval between R and R' in I, and let N_1 be a connected closed neighborhood of $\Pr_g RR'$ in Ω_g, such that (c) Fr $N_1 - \Pr_g I$ is polyhedral relative to $K(W)$. We may also suppose that (d) Fr $N_1 - \Pr_g I$ is a 2-manifold. (If not, add to N_1 a regular neighborhood of Fr $N_1 - \Pr_g I$ in an appropriate subdivision of $K(W)$.) We suppose that (e) $N_1 \cap D \subset$ Int D and (f) Fr $N_1 - \Pr_g I$ is in general position relative to D, in the usual sense; that is, the intersection of these two sets is the union of a finite collection of disjoint polygons, at each of which the two surfaces pierce one another. Finally, by a finite number of operations, each of which adds a set to N_1, or subtracts a set from N_1, we arrange so that (g) $N_1 \cap D$ is a 2-cell $d \subset$ Int D such that $d \cap$ Fr $N_1 =$ Bd d.

Consider $\tilde{N}_1 = \Pr_g^{-1} N_1 \subset V$. If N_1 lies in a sufficiently small neighborhood of $\Pr_g RR'$, then $\tilde{D} \cup \tilde{D}'$ will separate $\tilde{N}_1$ into exactly three connected sets. (The details here are straightforward.) Let C_1 be the closure of the component of $\tilde{N}_1 - (\tilde{D} \cup \tilde{D}')$ that contains Q. Now Fr C_1 is a tame 2-manifold, C_1 is a 3-manifold with boundary, with Bd C_1 = Fr C_1; and Bd C_1 is the union of a 2-cell $\tilde{D}_1 \subset$ Int $\tilde{D}$, a 2-cell $\tilde{D}_1' \subset$ Int $\tilde{D}'$ and a 2-manifold A with boundary which is a finite polyhedron relative to $K(\tilde{W})$. Here $g(\tilde{D}_1) = \tilde{D}_1'$, and $(\tilde{D}_1 \cup \tilde{D}_1') \cap A =$ Bd $\tilde{D}_1 \cup$ Bd $\tilde{D}_1' =$ Bd A, so that $g(A) = A$. Let C_0 be a geometrically round 3-cell neighborhood of Q in $\mathbf{R}^3$, lying in V, and thus intersecting I in exactly two points, and let $X =$ Int $C_0 - I$. Thus the fundamental group $\pi(X)$ is infinite cyclic. Evidently we can choose $\tilde{N}_1 \subset$ Int C_0. Thus we may suppose that (h) $\Pr_g$(Bd C_1 − Int C_0) lies in the union of a finite collection of polyhedral 2-cells, disjoint from one another and from $\Pr_g \tilde{D}_1$ ($= \Pr_g \tilde{D}_1'$).

At the outset, this holds because $C_1 \subset$ Int C_0. But we need to state (h) in such a way that it would be preserved by certain hypothetical operations to be defined presently. Now every compact connected 2-manifold B has Euler characteristic $\chi B < 2$. Therefore, finally, we may suppose that: (i) Subject to all the above conditions, C_1 is chosen so as to maximize χ Bd $\mathrm{Pr}_g\, C_1$.

We shall show that under these conditions C_1 is a 3-cell. For this, it will be sufficient to show that A is an annulus. It will then follow that Bd C_1 is a 2-sphere and since Bd C_1 is tame, C_1 is a 3-cell.

Let $P_0 \in$ Bd $\tilde{D}_1 \subset$ Bd A, and let p be a closed path, with base point P_0, which traverses Bd $\tilde{D}_1$ exactly once. Since I pierces $\tilde{D}_1$ at R, it follows that p generates $\pi(\mathrm{Int}\ C_0 - I, P_0)$. (More precisely, the equivalence class that contains p generates the group.) If p generates $\pi(\mathrm{Bd}\ C_1 - I, P_0)$ (in the same sense), then it follows that $\pi(A)$ is infinite cyclic and A is an annulus. If not, then there is a closed path q in Bd $C_1 - I$, with base point P_0, such that q is not equivalent, in $\pi(\mathrm{Bd}\ C_1 - I)$, to any power p^n of p. By (h), we may choose q as a closed path in $(\mathrm{Bd}\ C_1 - I) \cap$ Int C_0. Take n so that $q \simeq p^n$ in $\pi(\mathrm{Int}\ C_0 - I)$ and now regard qp^{-n} simply as a loop L (ignoring the distinguished base point). Then L is contractible in $\tilde{W} = V - I$ but not in Bd $C_1 - I$.

Consider now the 2-manifold $B = \mathrm{Pr}_g(\mathrm{Bd}\ C_1 - I)$ in the 3-manifold $W = \mathrm{Pr}_g\, \tilde{W}$. (We recall that $\tilde{W} = V - I$.) The mapping $\mathrm{Pr}_g|(\mathrm{Bd}\ C_1 - I)$ is a covering, and thus induces an injective homomorphism of the fundamental group. Thus the loop $\mathrm{Pr}_g\, L$ is contractible in W but not in B. When W is regarded as space, B forms a closed set. Evidently B is 2-sided in W, because $\mathrm{Pr}_g(C_1 - I)$ is a 3-manifold with boundary, lying in the 3-manifold W; B is its boundary and B is a polyhedron relative to $K(W)$. (See [**MGT**, Theorem 26.1, p. 191].) Thus the pair W, B satisfies the conditions of one form of the Loop Theorem of Papakyriakopoulos. (See [**M**, Theorem 3.5].) From this form of the Loop Theorem it follows that there is a polyhedral 2-cell Δ in W, with $\Delta \cap B =$ Bd Δ, such that Bd Δ is not contractible in B. Evidently we can move Bd Δ off of all the 2-cells mentioned in condition (h), preserving the stated properties of Δ.

Under these conditions for B and Δ, Bd Δ has an annular neighborhood in B [**MGT**, Theorem 28.19, p. 209]. When we "split B apart at Bd Δ", χB is unchanged; after the splitting, χ Bd Δ gets counted twice, but χ Bd $\Delta = 0$. Thus, if we split $B \cup \Delta$ apart at Δ, getting a new 2-manifold B', we have $\chi B' = \chi B + 2$.

We shall show that this is impossible. We recall that $R = \tilde{D}_1 \cap I$, $R' = g(R) = \tilde{D}_1' \cap I$. Thus $B \cup \mathrm{Pr}_g\, R = \mathrm{Pr}_g$ Bd C_1, and so $B \cup \mathrm{Pr}_g\, R$ separates each small neighborhood of $\mathrm{Pr}_g\, Q$ from each point of $\mathrm{Pr}_g(I - C_1)$ in the space Ω_g. From this it follows that $B' \cup \mathrm{Pr}_g\, R$ has the same separation property. This is fairly easy to see geometrically. For a detailed treatment, see [**MGT**, Theorem 30.3, p. 215]. (This theorem is stated for 3-manifolds. Therefore we delete $\mathrm{Pr}_g\, Q$ from Ω_g, apply the theorem, and then reinstate $\mathrm{Pr}_g\, Q$.) Let N be the closure of the component of $\Omega_g - (B' \cup \mathrm{Pr}_g\, R)$ that contains $\mathrm{Pr}_g\, Q$ and let $C_1' = \mathrm{Pr}_g^{-1} N$. Then C_1' satisfies all the conditions that were stated for C_1. (In verifying (h), note that in passing from B to B', we have, at most, added two more 2-cells to the collection that we already had.) All this is impossible because C_1 was supposed to be chosen so as to maximize χB.

It remains to show that given a neighborhood $N(Q)$ of Q, C_1 can be chosen so as to lie in $N(Q)$. In the above proof, we know, at least, that $C_1 \subset V$. Given $N(Q)$, we can choose a round 3-cell C_0 with center at Q such that $V' = \text{Int } C_0 \cup g(\text{Int } C_0) \subset N(Q)$. Now V', Q, and $g|V'$ satisfy all the conditions for V, Q, and g in the lemma. Therefore we can choose C_1 so that $C_1 \subset V' \subset N(Q)$. Lemma 5.7.1 now follows.

In the following discussion, the notation of Lemma 5.7.1 and of the paragraph preceding it, will be regarded as conventions, but symbols used in the *proof* of the lemma may be used in different senses below.

Let S be a polyhedral solid torus (in a triangulated 3-manifold), let $T = \text{Bd } S$, and let J and J_i be polygons in T. As in §4 of **[M]**, J is *latitudinal* (in S) if J bounds a 2-cell in S but not in T; and J_i is *longitudinal* if J_i carries a generator of the 1-dimensional homology group $H_1(S)$ (with integer coefficients). Theorem 4.6 of **[M]** asserts that if, also, J and J_i are in general position, then J_i crosses J algebraically once. (Conversely, if J is latitudinal, J and J_i are in general position, and J_i crosses J algebraically once, then J_i is longitudinal.) Theorem 4.7 of **[M]** asserts that if J is latitudinal and $J_1, J_2, \ldots, J_m$ are longitudinal and disjoint, then there is a PL homeomorphism $\phi: S \leftrightarrow S$ such that $\phi(J)$ intersects each J_i in exactly one point, which is a "true crossing point".

For the case in which S is a solid Klein Bottle, we define the terms *latitudinal* and *longitudinal* in the same way. In this case Theorems 4.6 and 4.7 of **[M]** are still true and their proofs are essentially the same.

LEMMA 5.7.2. *Under the conditions of Lemma* 5.7.1, *let* D_1 *and* $D_1' = g(D_1)$ *be disjoint* 2-*cells in* Bd C_1, *with boundaries which are polyhedra relative to* $K(\tilde{W})$, *and containing in their interiors the points* R *and* R' *of* $I \cap \text{Bd } C_1$. *Let* A_1 *be the annulus in* Bd C_1 *such that* $\text{Bd } A_1 = \text{Bd } D_1 \cup \text{Bd } D_1'$. *Let* A_2 *be a* g-*invariant polyhedral annulus in* C_1 *such that* (5) $\text{Bd } A_2 = \text{Bd } A_1 = A_2 \cap \text{Bd } C_1$ *and* (6) $A_2 \cap I = \varnothing$. *Let* $\tilde{T} = A_1 \cup A_2$, *let* $\tilde{S}$ *be the closure of the bounded component of* $\mathbf{R}^3 - \tilde{T}$, *let* $T = \text{Pr}_g\, \tilde{T}$, *and let* $S = \text{Pr}_g\, \tilde{S}$. *If* C_1 *lies in a sufficiently small neighborhood of* Q, *then*

(7) $\tilde{S}$ *is a solid torus.*

(8) S *is a solid Klein Bottle.*

(9) *There is a* 2-*cell* $E_1 \subset \tilde{S}$ *such that* $E_1 \cap g(E_1) = \varnothing$, $\text{Bd } E_1$ *is latitudinal in* $\tilde{S}$, $\text{Int } E_1 \subset \text{Int } \tilde{S}$, *and* E_1 *intersects* $\text{Bd } A_1$ *in exactly two points, which are "true crossing points"*.

PROOF OF LEMMA. Since I is a linear interval, it follows that there is a tame 2-cell d such that $d \cap I$ is an arc in Bd d, containing Q in its interior. Obviously $d - I$ is locally tame relative to $K(\tilde{W})$. Thus $d - I$ is tame relative to $K(\tilde{W})$, and so we may assume that $d - I$ is a polyhedron relative to $K(\tilde{W})$.

We now take C_1 in a sufficiently small neighborhood of Q so that $C_1 \cap \text{Bd } d \subset \text{Int}(d \cap I)$. Thus Bd d pierces Bd C_1 at R and R', and any generators of the 1-dimensional homology groups $H_1(\text{Bd } d)$ and $H_1(\text{Bd } D)$ (with integer coefficients) link one another with linking number ± 1.

Now consider $\tilde{S}$, $\tilde{T}$ as in the lemma. We move Int d into general position relative to $\tilde{T}$ in the usual sense; and now we choose d (subject to all the above conditions) so as to minimize the number of components of $d \cap \tilde{T}$.

Let J be a polygon which is a component of $d \cap \tilde{T}$, and let d_J be the 2-cell in d such that Bd $d_J = J$. Then J does not bound a 2-cell d_J' in $\tilde{T}$, because if so we could replace d_J by d_J' in d and move the resulting 2-cell off of $\tilde{T}$ in the neighborhood of d_J'. This is impossible because it reduces the number of components of $d \cap \tilde{T}$. If Int $d_J \subset \mathbf{R}^3 - \tilde{S}$, then J is contractible in $\mathbf{R}^3$ − Int $\tilde{S}$ but does not bound a 2-cell in $\tilde{T}$. Theorem 4.9 of **[M]** asserts that under these conditions, J is longitudinal in $\tilde{S}$. Thus any small regular neighborhood of $\tilde{S} \cup d_J$ is a 3-cell in $\mathbf{R}^3$ − Bd d containing $\tilde{S}$; and this is impossible, because a 1-cycle on $\tilde{S}$ links a 1-cycle on Bd d with linking number ± 1.

Thus (a) Int $d_J \subset$ Int S. We know that Bd d_J does not bound a 2-cell in $\tilde{T}$, and from this it follows that (b) Bd d_J is not contractible in $\tilde{T}$.

PROOF. If Bd d_J is contractible in $\tilde{T}$, then a cycle Z_J^1 which generates H_1(Bd d_J) bounds on $\tilde{T}$, from which we can easily show that Bd d_J separates $\tilde{T}$. Thus $\tilde{T}$ is the union of two connected 2-manifolds $\tilde{T}_1$, $\tilde{T}_2$ with boundary, with Bd $\tilde{T}_1$ = Bd $\tilde{T}_2$ = $\tilde{T}_1 \cap \tilde{T}_2$ = Bd d_J. Each $\tilde{T}_i$ is thus a 2-cell with (possible) handles, and since $\tilde{T}$ is a torus–i.e., a 2-sphere with only one handle–it follows that one of the sets $\tilde{T}_i$ is a 2-cell, which is false. Therefore (b) Bd d_J is not contractible in $\tilde{T}$.

Let $J' = \mathrm{Pr}_g\, J$. Thus J' is contractible in $S = \mathrm{Pr}_g\, \tilde{S}$. Since $g|\tilde{T}\colon \tilde{T} \to T \subset \Omega_g$ is a 2-fold covering, J' is not contractible in $T = \mathrm{Pr}_g\, \tilde{T}$ = Bd S. By the Loop Theorem it follows that there is a polyhedral 2-cell $\Delta \subset S$, with $\Delta \cap T$ = Bd Δ, such that Bd Δ is not contractible in T. Now $\mathrm{Pr}_g^{-1}\, \Delta$ is the union of two disjoint 2-cells E_1, E_1', with $E_1' = g(E_1)$, and Bd E_1 and Bd E_1' are latitudinal in $\tilde{S}$. Since Bd D_1 is longitudinal in $\tilde{S}$, it follows that Bd D_1 crosses Bd E_1 algebraically once. Therefore Pr_g Bd D_1 crosses Bd Δ algebraically once, so that Pr_g Bd D_1 is longitudinal in S.

Of course, $E_1 \cup E_1'$ decomposes $\tilde{S}$ into two 3-cells each of which is mapped onto S by Pr_g. Thus $\tilde{S}$ is a solid torus. Since g reverses orientation, S is a solid Klein Bottle rather than a solid torus.

By the results cited just before Lemma 5.7.2, Δ can be chosen in such a way that Bd Δ crosses Pr_g Bd D_1 exactly once, in a true crossing point. Now either of the components of $\mathrm{Pr}_g^{-1}\, \Delta$ can be taken as the E_1 of the conclusion of Lemma 5.7.2.

LEMMA 5.7.3. *Let* C_1 *and* C_2 *be* 2-*cells satisfying the conditions of Lemmas* 5.7.1 *and* 5.7.2, *such that* $C_2 \subset$ Int C_1. *Then there is an annulus* $\tilde{B} \subset \mathrm{Cl}(C_1 - C_2)$ *such that*

(10) $\tilde{B}$ *is tame in* V,

(11) $g(\tilde{B}) = \tilde{B}$,

(12) Int $\tilde{B} \subset$ Int $C_1 - C_2$,

(13) Bd $\tilde{B}$ *is the union of two* 1-*spheres* $J_1 \subset$ Bd C_1 *and* $J_2 \subset$ Bd C_2 *and*

(14) $I \cap \mathrm{Cl}(C_1 - C_2) \subset \tilde{B}$.

(Here (10) means merely what it says: $\tilde{B} - I$ is not necessarily a polyhedron relative to $K(\tilde{W})$.)

PROOF OF LEMMA. As before, let $I \cap$ Bd $C_1 = \{R, R'\}$; and let $I \cap$ Bd C_2 = $\{S, S'\}$, where RS and $R'S'$ are the components of $I \cap \mathrm{Cl}(C_1 - C_2)$. Let $\mathbf{B}^2$ be the unit ball in $\mathbf{R}^2$ with center (0, 0) and consider the cylinder $\mathbf{B}^2 \times [0, 1]$. We assert that C_1 and C_2 can be chosen in such a way that there is a homeomorphism ϕ: $\mathbf{B}^2 \times [0, 1] \leftrightarrow C_3 \subset C_1 - \text{Int } C_2$, such that (a) $\phi(\mathbf{B}^2 \times 0) = D_1 = C_3 \cap \text{Bd } C_1$, (b) $\phi(\mathbf{B}^2 \times 1) = D_2 = C_3 \cap \text{Bd } C_2$, (c) $\phi((0, 0) \times [0, 1]) = RS$, (d) $\text{Bd } C_3 - \{R, S\}$ is a polyhedron relative to $K(\tilde{W})$, and (e) $C_3 \cap g(C_3) = \varnothing$. Such a ϕ can be constructed as follows. Let E_R and E_S be the planes through R and S orthogonal to I. Since Bd C_1 and Bd C_2 are tame, we can move neighborhoods of R and S, in Bd C_1 and Bd C_2, into E_R and E_S respectively, by a homeomorphism ψ: $V \leftrightarrow V$ which is close to the identity and which is the identity on I. Now RS is the axis of symmetry of a thin cylinder C with its bases in E_R and E_S. Let $C' = \psi^{-1}(C)$. Now move the 1-spheres $\text{Bd}(C' \cap \text{Bd } C_i)$ onto polygons by a homeomorphism $V \leftrightarrow V$, $\text{Bd } C_i \leftrightarrow \text{Bd } C_i$. By Theorem 5.2, we can move $\text{Bd } C' - \{R, S\}$ onto a polyhedron by a homeomorphism ρ: $V \leftrightarrow V$ which is the identity on $\text{Bd } C_1 \cup \text{Bd } C_2$ and which differs from the identity only in an arbitrarily small neighborhood of $\text{Cl}[\text{Bd } C' - (\text{Bd } C_1 \cup \text{Bd } C_2)]$. Let $C_3 = \rho(C')$. Then C_3 satisfies conditions (a)–(d) of the lemma. We know that $RS \cap g(RS) = \varnothing$ because $g(RS) = R'S'$. Therefore, to ensure that C_3 satisfies (e), we merely choose C in a sufficiently small neighborhood of RS.

Let $A_1 = \text{Cl}[\text{Bd } C_1 - (C_3 \cup g(C_3))]$, so that A_1 is an annulus. Thus $\text{Bd } C_1 = A_1 \cup D_1 \cup D_1'$, where $D_1 = \text{Bd } C_1 \cap \text{Bd } C_3$ and $D_1' = \text{Bd } C_1 \cap \text{Bd } g(C_3)$. Let $A_2 = \text{Cl}[\text{Int } C_1 \cap \text{Bd}(C_3 \cup C_2 \cup g(C_3))]$, so that C_1, A_1, and A_2 satisfy the conditions of Lemma 5.7.2. Let E_1 be as in the conclusion of Lemma 5.7.2. Now A_2 is the union of the three annuli $A_2 \cap C_3$, $A_2 \cap C_2$, and $A_2 \cap g(C_3)$. As in the proof of Lemma 5.7.2, we can "straighten out" $\text{Bd } E_1 \cap A_2$ in such a way that $\text{Bd } E_1$ intersects $\text{Bd}(A_2 \cap C_2)$ in exactly two points, which are true crossing points.

Consider the set $\text{Pr}_g(E_1 \cup C_3) \subset \Omega_g - \text{Pr}_g\, Q$. This set is tame. Therefore it is a polyhedron relative to some triangulation K of the 3-manifold $\Omega_g - \text{Pr}_g\, Q$. The set $\text{Pr}_g\, E_1$ intersects $\text{Bd Pr}_g\, C_3$ in the union of two disjoint broken lines $b = \text{Pr}_g(E_1 \cap \text{Bd } C_3)$, and $b' = \text{Pr}_g(E_1 \cap g(C_3))$; and each of the broken lines b, b' have end points in $\text{Pr}_g \text{ Bd } C_1$ and $\text{Pr}_g \text{ Bd } C_2$. Since $\text{Pr}_g(I \cap C_3)$ is "unknotted in $\text{Pr}_g\, C_3$", in the obvious sense (or see [MGT, p. 134]), it follows by an easy construction that there is a 2-cell $d \subset \text{Pr}_g\, C_3$, containing $\text{Pr}_g(I \cap C_3)$, such that $d \cap \text{Bd } C_3 = \text{Bd } d$ and such that Bd d is the union of b, b', an arc in $\text{Pr}_g \text{ Bd } C_1$, and an arc in $\text{Pr}_g \text{ Bd } C_2$. Thus d is "untwisted in $\text{Pr}_g\, C_3$", and the set $B = \text{Pr}_g\, E_1 \cup d$ is an annulus (rather than a Möbius band). Let $\tilde{B} = \text{Pr}_g^{-1} B$. Since B is tame in $\Omega_g - \text{Pr}_g\, Q$, it follows that $\tilde{B}$ is locally tame in $V - Q$, and so $\tilde{B}$ is tame in V. Thus $\tilde{B}$ satisfies all the conditions of Lemma 5.7.3.

Let N^2 be a triangulated 2-manifold and let h be a homeomorphism $N^2 \leftrightarrow N^2$. If there is a polyhedral 2-cell d such that $h|(N^2 - d)$ is the identity, then h is *cellular*. Suppose also that N lies in Bd N^3 where N^3 is a 3-manifold with boundary. Then there is a 3-cell $C^3 \subset N^3$ such that $d = \text{Bd } C^3 \cap \text{Bd } N^3$; C^3 can be chosen so as to lie in any preassigned closed neighborhood of Int d; and now h can be extended so as to give a homeomorphism $N^3 \leftrightarrow N^3$ which differs from the identity only in $\text{Int } C^3 \cup \text{Int } d$. (To get the extension, we express C^3 as the join of d and a point.) A similar conclusion holds if N^2 is tame in a 3-manifold N^3; we take 3-cells C_1^3,

C_2^3, "close to d" such that $C_1^3 \cap C_2^3 = d = C_i^3 \cap N^2$.

LEMMA 5.7.4. *Let J be a* 1*-sphere in a triangulated* 2*-manifold N^2. Then J is tame. In fact, given $Q \in J$, J can be moved onto a polyhedron by a finite sequence of cellular homeomorphisms each of which leaves Q fixed.*

INDICATION OF PROOF. Theorem 10.7 on p. 73 of [**MGT**] asserts that in $\mathbf{R}^2$ every topological linear graph M without end points is tame. The same proof applies, to give the same conclusion, in an arbitrary triangulated 2-manifold. In the proof, we take a triangulation of M, so that certain points and arcs become "vertices" and "edges" of M. We then move M onto a polyhedron by a sequence of cellular homeomorphisms each of which leaves every "vertex" of M fixed. (In [**MGT**], it was not noted that these homeomorphisms are cellular, but they are, simply by construction.) Since any point Q of J is a vertex in some triangulation of J, the proof just described is also a proof of Lemma 5.7.4.

By a *square diagram* of a projective plane N^2 we mean a mapping ψ: $[0, 1]^2 \twoheadrightarrow N^2$ which identifies antipodal points of $\mathrm{Bd}[0, 1]^2$ and is a homeomorphism elsewhere. N^2 has a triangulation which is the image of a rectilinear triangulation of $[0, 1]^2$. Polyhedra in N^2 are defined relative to such a triangulation.

LEMMA 5.7.5. *Let N^2 be a projective plane, let J be a* 1*-sphere in N^2, and suppose that J does not bound a* 2*-cell in N^2. Then*

(1) *N^2 has a square diagram ρ: $[0, 1]^2 \twoheadrightarrow N^2$ such that $J = \rho(\mathrm{Bd}[0, 1]^2)$.*

(2) *Let ψ: $[0, 1]^2 \twoheadrightarrow N^2$ be any square diagram of N^2 and let Q be a point of $J \cap \psi(\mathrm{Bd}[0, 1]^2)$. Then J can be moved onto $\psi(\mathrm{Bd}[0, 1]^2)$ by a finite sequence of cellular homeomorphisms each of which leaves Q fixed.*

Here (1) means that a 1-sphere can be imbedded in a projective plane in essentially only two ways. It is easy to show that (2) implies (1). Given a square diagram ψ for N^2, J must intersect $\psi(\mathrm{Bd}[0, 1]^2)$, since otherwise J would bound a 2-cell in N^2. Let h be a homeomorphism as in (2), and let $\rho = h^{-1}(\psi)$.

It remains to prove (2). First, by a cellular homeomorphism $N^2 \leftrightarrow N^2$, leaving Q fixed, we move J onto a polygon J_1. By another such homeomorphism, we move J_1 onto a polygon J_2 which is in "almost general position", in the sense that the set $J_2 \cap \psi(\mathrm{Bd}[0, 1]^2)$ is finite and each of its points, except perhaps for Q, is a true crossing point of J and $\psi(\mathrm{Bd}[0, 1]^2)$. Let $K = \psi^{-1}(J_2) \subset [0, 1]^2$. Let $b_1, b_2, \ldots, b_m$ be the closures of the components of $K \cap \mathrm{Int}[0, 1]^2$. No set b_i can be a polygon, because if so, $b_i = K$ and J_2 bounds a 2-cell in N^2, which is false. Therefore each b_i is a broken line with its end points in $\mathrm{Bd}[0, 1]^2$.

Case 1. Suppose that some b_i has end points v, v' which are antipodal. Then $b_i = K$ and $\{v, v'\} = \psi^{-1}(Q)$. Let b' be either of the two arcs in $\mathrm{Bd}[0, 1]^2$ between v and v'. By two cellular homeomorphisms $N^2 \leftrightarrow N^2$, leaving Q fixed, we can move $\psi(b_i)$ onto $\psi(b')$, and now we are done, because $\psi(b') = \psi(\mathrm{Bd}[0, 1]^2)$. (First we move "half of b_i" onto "half of b'" by a homeomorphism of the above type and then we move "the rest of b_i" onto "the rest of b'".)

Case 2. Suppose that no set b_i has antipodal end points v, v'. Let b' be the arc in $\mathrm{Bd}[0, 1]^2$ between v and v' which is *short*, in the sense that it does not intersect its image under the antipodal mapping. Now $b_i \cup b'$ is the boundary of a 2-cell $d_i \subset [0, 1]^2$. Evidently some b_i is *outermost* in $[0, 1]^2$, in the sense that d_i contains no set $b_j \neq b_i$. Given such a b_i, suppose that $\{v, v'\} \cap \psi^{-1}(Q) = \varnothing$, so that $\psi(v)$ and $\psi(v')$ are crossing points of J_2 and $\psi(\mathrm{Bd}[0, 1]^2)$. Thus $\psi(b_i)$ can be moved across $\psi(\mathrm{Bd}[0, 1]^2)$ by a cellular homeomorphism $N^2 \leftrightarrow N^2$ leaving Q fixed. This reduces m. If $\psi(v) = Q$ or $\psi(v') = Q$, then we can move $\psi(b_i) - Q$ across $\psi(\mathrm{Bd}[0, 1]^2)$ by a homeomorphism of the same type, and this also reduces m. Thus, in a finite number of steps, we get Case 1 and the lemma follows.

Let C be a set of points in a metric space. Then δC is the supremum of the distances $d(P, Q)$ where $P, Q \in C$. (Thus we may have $\delta C = \infty$.)

LEMMA 5.7.6. *Under the conditions of Lemma* 5.7.1, *there is a sequence* $C_1, C_2, \ldots$ *of* 3-*cells and a sequence* $\tilde{B}_1, \tilde{B}_2, \ldots$ *of annuli, such that* (a) *for each* i, C_i, C_{i+1}, *and* $\tilde{B}_i$ *satisfy the conditions for* C_1, C_2, *and* $\tilde{B}$ *in Lemma* 5.7.3, (b) $\mathrm{Lim}_{i\to\infty} \delta C_i = 0$, (c) *the set* $D = \bigcup_{i=1}^{\infty} \tilde{B}_i \cup \{Q\}$ *is a* 2-*cell, and* (d) D *is tame.*

Here (d) is equivalent to the statement that C_1 is the union of two 3-cells E^+ and E^-, with $E^+ \cap E^- = \mathrm{Bd}\, E^+ \cap \mathrm{Bd}\, E^- = D$.

PROOF OF LEMMA. By repeated applications of Lemma 5.7.3, we get $C_1, C_2, \ldots$ and $\tilde{B}'_1, \tilde{B}'_2, \ldots$ satisfying the conditions for C_i and $\tilde{B}_i$ in (a) and (b). Next we need to show that the boundaries of successive annuli $\tilde{B}'_i$ can be made to coincide, so that (c) is satisfied. For each i, let $B'_i = \mathrm{Pr}_g\, \tilde{B}'_i$ and let $T_i^2 = \mathrm{Pr}_g\, \mathrm{Bd}\, C_i$. Let $Q_i = \mathrm{Pr}_g(I \cap \mathrm{Bd}\, C_i)$, so that for each $i > 1$, $Q_i \in B'_{i-1} \cap B'_i \subset T_i^2$.

$$J_i = \mathrm{Bd}\, B'_{i-1} \cap T_i^2, \qquad J'_i = \mathrm{Bd}\, B'_i \cap T_i^2.$$

Evidently neither J_i nor J'_i bounds a 2-cell in T_i^2. Therefore, by conclusion (1) of Lemma 5.7.5, T_i^2 has a square diagram ψ such that $J_i = \psi(\mathrm{Bd}[0, 1]^2)$. By (2) of Lemma 5.7.5, J'_i can be moved onto J_i by a cellular homeomorphism $T_i^2 \leftrightarrow T_i^2$ leaving Q_i fixed. We need to extend this to get a homeomorphism h: $\mathrm{Pr}_g\, C_i \leftrightarrow \mathrm{Pr}_g\, C_i$ such that $h|\mathrm{Pr}_g\, C_{i+1}$ and $h|\mathrm{Pr}_g(C_i \cap I)$ are identity mappings. See the discussion just before Lemma 5.7.4. Given a cellular homeomorphism h_j: $T_i^2 \leftrightarrow T_i^2$ leaving Q_i fixed, let d be the 2-cell on which h_j may not be the identity. If $Q_i \notin \mathrm{Int}\, d$, then we take C^3 as usual, "close to d" and not intersecting $\mathrm{Pr}_g\, C_{i+1}$ or $\mathrm{Pr}_g\, I$ (except perhaps at Q_i if $Q_i \in \mathrm{Bd}\, d$). If $Q_i \in \mathrm{Int}\, d$, then we take C^3 in such a way that $\mathrm{Pr}_g\, I$ pierces $\mathrm{Bd}\, C^3$ at exactly two points Q_i and Q' and intersects $\mathrm{Bd}\, C^3$ nowhere else; and we express C^3 as the join of d and Q' in such a way that $C^3 \cap \mathrm{Pr}_g\, I$ is the join of Q_i and Q'. Since $h(Q_i) = Q_i$, the standard extension of $h_j|d$ to C^3 is the identity on $C^3 \cap \mathrm{Pr}_g\, I$, as desired.

Now let $B_i = h(B'_i)$, $\tilde{B}_i = \mathrm{Pr}_g^{-1}\, B_i$. Then $\tilde{B}_1, \tilde{B}_2, \ldots$ satisfies (c). Trivially, (b) is still satisfied. And so also is (a): the extensions of the homeomorphisms h_j were defined so as to ensure that $I \cap \mathrm{Cl}(C_i - C_{i+1}) \subset \tilde{B}_i$.

It remains to prove (d). Now $\tilde{B}_1 \subset \mathrm{Bd}\, C_1$ decomposes $\mathrm{Bd}\, C_1$ into two 2-cells. Let D_1 be one of these. There is then a 2-cell $D_2 \subset \mathrm{Bd}\, C_2$ such that $D_1 \cup \tilde{B}_1 \cup D_2$ is the boundary of a 3-cell D_1^3, disjoint from $\mathrm{Int}\, C_2$. Recursively, we get a sequence

$D_1^3, D_2^3, \ldots$ of 3-cells with $D_i^3 \cap D_{i+1}^3 = D_{i+1} \subset \text{Bd}\, C_{i+1}$ for each i. By an obvious construction, the set

$$E^+ = \bigcup_i D_i^3 \cup \{Q\}$$

is a 3-cell. (For a very similar construction, see [**MGT**, pp. 140–141].) Starting with $D_1' = \text{Cl}(\text{Bd}\, C_1 - D_1)$, we get a similar sequence of 3-cells whose union together with Q is a 3-cell E^-. Thus

$$C_1 = E^+ \cup E^-, \qquad E^+ \cap E^- = D,$$

and so D is locally tame at every point of Int D. Since $\tilde{B}_1$ is tame, D is locally tame at every point of Bd $D \subset \text{Bd}\, \tilde{B}_1$. Therefore D is tame, and (d) holds.

LEMMA 5.7.7. *Under the conditions of Lemma* 5.7.6, $\text{Pr}_g\, C_1$ *has a star-triangulation* $K_1 = \text{St}\, \text{Pr}_g\, Q$ *in which* $\text{Pr}_g(C_1 \cap I)$ *forms an edge.*

PROOF OF LEMMA. Consider the homeomorphism $g|D$: $D \leftrightarrow D$, which reverses orientation and has Q as its only fixed point. Thus $g|D$ is conjugate to the antipodal mapping $\mathbf{B}^2 \leftrightarrow \mathbf{B}^2$. Also, $g(D \cap I) = D \cap I$. It follows that D has a star-triangulation $K(D) = \text{St}\, Q$ such that $D \cap I$ is the union of two edges of $K(D)$ and such that $g|D$ is simplicial relative to $K(D)$. Evidently $K(D)$ can be extended so as to give a star-triangulation $K(E^+) = \text{St}\, Q$ of the 3-cell E^+ described in the proof of the preceding lemma. Now let

$$K_1 = \text{Pr}_g\, K(E^+) = \{\text{Pr}_g\, \sigma | \sigma \in K(E^+)\}.$$

Then K_1 is as desired in Lemma 5.7.7.

From Lemma 5.7.7, Theorem 5.7 follows immediately.

Let N^3 be a 3-manifold with boundary and let L be a set of points in N^3. If N^3 has a triangulation relative to which L is a polyhedron, then L is *tame* (in N^3). (In the case in which N^3 is triangulated and Bd $N^3 = \varnothing$, this definition of *tame* is equivalent to the standard definition.) Suppose that for every point P of L there is a closed neighborhood N_P of P in $L \cup \text{Bd}\, N^3$, an open neighborhood U_P of N_P in N^3, and a homeomorphism h: $U_P \to \text{Int}\, N^3$, such that $h(N_P)$ is tame in Int N^3. Then L is *locally tame* (in N^3).

THEOREM 5.8. *Let* N^3 *be a compact* 3-*manifold with boundary and let* L *be a set of points in* N^3 *such that* L *is locally tame. Then* L *is tame. If* L *is compact, then* N^3 *has a triangulation in which* L *forms a subcomplex.*

PROOF. N^3 lies in a 3-manifold M^3 in which Bd N^3 is the frontier of N^3. (For example, we may adjoin to N^3 the product Bd $N^3 \times [0, 1)$, where $[0, 1)$ is the half-open interval.) Our hypothesis implies that $L \cup \text{Bd}\, N^3$ is locally tame relative to any triangulation $K(M^3)$ of M^3. By the local tame imbedding theorem, $L \cup \text{Bd}\, N^3$ is tame in M^3. Thus there is a homeomorphism g: $M^3 \leftrightarrow M^3$ such that $g(L \cup \text{Bd}\, N^3)$ is polyhedral relative to $K(M^3)$. Since Bd N^3 is compact, $K(M^3)$ has a subdivision $K'(M^3)$ in which $g(N^3)$ forms a subcomplex $K'(g(N^3))$ and $g(L \cup \text{Bd}\, N^3)$ is polyhedral relative to $K'(g(N^3))$. Mapping back by $(g|N^3)^{-1}$, we get a triangulation $K(N^3)$ relative to which $L \cup \text{Bd}\, N^3$ is polyhedral. If L is compact, then $K(N^3)$ has a subdivision in which L forms a subcomplex.

6. Proof of Theorem 1.4: Conclusion. We resume the discussion in §4. We have Ω_{r-q} and $K(\Omega_{r-q})$, satisfying the conditions of Theorem 4.4. We have

$$n = p_1 p_2 \cdots p_r = 2^q p_{q+1} p_{q+2} \cdots p_r.$$

For $r - q \leqslant i < r$, we have $m_i = p_1 p_2 \cdots p_{r-i} = 2^{r-i}$, and for $i = r$ we also get the "right answer" $m_r = 1$. Thus $m_{r-q+1} = 2^{q-1} = k$ and f^k induces a homeomorphism

$$f_*^k: \Omega_{r-q} \leftrightarrow \Omega_{r-q}$$

of period 2. As before, let Ω^*_{r-q+1} be the orbit space of f_*^k and let Pr^*_{r-q+1} be the projection $\Omega_{r-q} \to \Omega^*_{r-q+1}$. As before, Ω^*_{r-q+1} and Ω_{r-q+1} are essentially indistinguishable.

THEOREM 6.1. *Ω_{r-q+1} has a triangulation $K(\Omega_{r-q+1})$ in which each set $\mathrm{Pr}_{r-q+1} F_{n_s}$ forms a subcomplex and in which each simplex intersects $\bigcup_s \mathrm{Pr}_{r-q+1} F_{n_s}$ in a simplex.*

PROOF. As for $i < r - q$, let G_{r-q} be the fixed-point set of f_*^k. Thus $G_{r-q} \subset \mathrm{Pr}_{r-q} F_{n/2}$ and G_{r-q} is a manifold. The components G of G_{r-q} may be manifolds of any dimension from 0 to 2; they need not all be of the same dimension. Hereafter G will always be a component of G_{r-q}.

LEMMA 6.1.1. *Suppose that G is a 1-sphere. Then there is a component F of $F_{n/2}$ such that F is a 1-sphere and $\mathrm{Pr}_{r-q} F = G$.*

PROOF OF LEMMA. Let F be a component of $F_{n/2}$ that intersects $\mathrm{Pr}_{r-q}^{-1} G$. Then F is a manifold of dimension 1 or 2. We know that the orbit space Ω_{r-q+1} is homeomorphic to Ω^*_{r-q+1}. Now Ω_{r-q} is locally Euclidean and $f_*^k|(\Omega_{r-q} - G_{r-q+1})$ is a covering. Let W^* be an open set in Ω^*_{r-q+1}, containing $\mathrm{Pr}^*_{r-q+1} G$ such that

$$W^* \cap \mathrm{Pr}^*_{r-q+1} G_{r-q+1} = \mathrm{Pr}^*_{r-q+1} G.$$

Then W^* is locally Euclidean except perhaps at the points of a 1-dimensional set (namely, $\mathrm{Pr}^*_{r-q+1} G$). It follows that the corresponding set $W \subset \Omega_{r-q+1}$ is locally Euclidean except perhaps at the points of a 1-dimensional set.

But Ω_{r-q+1} can also be formed in the following way. We rearrange the prime factors of n, writing $n = 2^{q-1} p_{q+1} p_{q+2} \cdots p_r 2$. Using this order, we get a new sequence $\Omega'_1, \Omega'_2, \ldots, \Omega'_{r-q}, \Omega'_{r-q+1}$, where Ω'_1 is the orbit space of $f_{n/2}$ and the rest of the sequence is formed as in §4. Let Pr'_i be the projection $M \to \Omega'_i$.

Now suppose that F is a 2-manifold. Then $f^{n/2}$ reverses local orientation at each point of F and no point of $\mathrm{Pr}'_1 F$ has a Euclidean neighborhood in Ω'_1. In Theorem 4.1 and the preceding discussion, we did not use the hypothesis that the p_i's were arranged in order of magnitude. Therefore Theorem 4.1 implies to the formation of the new orbit space sequence, and so, at every stage, Ω'_{i+1} is the orbit space of Ω'_i under a homeomorphism whose fixed-point set is at most 1-dimensional, such that the projection $\Omega'_i \to \Omega'_{i+1}$ is a local homeomorphism elsewhere. Therefore every neighborhood of every point of $\mathrm{Pr}'_{r-q+1} F$ in Ω'_{r-q+1} contains a 2-dimensional set none of whose points have open Euclidean neighborhoods in Ω'_{p-q+1}; the latter is impossible, because $\Omega'_{r-q+1} = \Omega_{r-q+1}$. Therefore F is a 1-manifold, and since F is connected, F is a 1-sphere. This holds for every component of $F_{n/2}$ that intersects $\mathrm{Pr}_{r-q}^{-1} G$.

Since G is connected, G lies in a single component C of $\mathrm{Pr}_{r-q} F_{n/2}$. By (3) of Theorem 4.3, $C = \mathrm{Pr}_{r-q} C'$, for some component C' of $F_{n/2}$. Let $F = C'$. Then F is a 1-sphere and $G = \mathrm{Pr}_{r-q} F$.

We resume the proof of Theorem 6.1. For $j = 1, 2$, let M^j be the union of all j-dimensional components of all sets $\mathrm{Pr}_{r-q} F_{n_s} \subset \Omega_{r-q}$. Let $g = f_*^k$. By Theorems 4.3 and 4.4, Ω_{r-q}, M^1, M^2, and g satisfy all the conditions for M^3, M^2, M^1, and h in §5. We shall build up the desired triangulation of Ω_{r-q+1} in two steps as follows.

Step 1. Let G^0 be the set of all isolated points of G_{r-q}. If $G^0 = \varnothing$, proceed immediately to Step 2. If not, let $P_i \in G^0$. For $P_i \in M^2 - M^1$, let C_i be the C of Theorem 5.5; for $P_i \in M^1 \cap M^2$, let C_i be the C of Theorem 5.6; and for $P_i \in M^1 - M^2$, let C_i be the C of Theorem 5.7. In each of these cases, let $K(C_i)$ be the star-triangulation of $\mathrm{Pr}^*_{r-q+1} C_i$ given by the corresponding theorem so that each $K(C_i)$ has the form $\mathrm{St}\,\mathrm{Pr}^*_{r-q+1} P_i$.

Finally, for $P_i \notin M^1 \cup M^2$, we use Theorem 1.3 to get the analogous C_i and $K(C_i)$. (Note that this is the most difficult case; it is the one that requires Theorem 1.2 (Rubinstein).)

We choose the sets C_i in such a way that they are disjoint.

Step 2. Let

$$N^3 = \Omega^*_{r-q+1} - \bigcup_i \mathrm{Pr}^*_{r-q+1} \operatorname{Int} C_i,$$

and let

$$L = N^3 \cap \bigcup_s \mathrm{Pr}^*_{r-q+1} F_{n_s}.$$

LEMMA 6.1.2. *N^3 is a 3-manifold with boundary, and L is tame in N^3 in the sense of Theorem* 5.8.

PROOF OF LEMMA. Let G be a 1-dimensional component of G_{r-q}. Then $G \cap \bigcup_i C_i = \varnothing$ and g preserves local orientation at each point of G. By Lemma 6.1.1, G is a component of $\mathrm{Pr}_{r-q} F_{n/2}$. By Theorem 4.3 it follows that G is tame. By Theorem 2.1, Ω^*_{r-q+1} is locally Euclidean at each point of $G' = \mathrm{Pr}^*_{r-q+1} G$, and G' is tame in a neighborhood of G'. Since G is a 1-dimensional component of $\mathrm{Pr}_{r-q} F_{n/2}$, it follows that G is a component of $\mathrm{Pr}_{r-q} \bigcup_s F_{n_s}$ and hence that G is open in $\mathrm{Pr}_{r-q} \bigcup_s F_{n_s}$. Therefore G' is open in L, and L is locally tame at each point of G'.

Let G be a 2-dimensional component of G_{r-q}. Then each point P of $G' = \mathrm{Pr}^*_{r-q+1} G$ has a 3-cell neighborhood V in N^3 such that $L \cap V$ is tame. The point is that G is tame, being a component of $\mathrm{Pr}_{r-q} F_{n/2}$, and g reverses local orientation at each point P of G. Thus, in small neighborhoods of P, N^3 looks simply like a subspace of $M - \bigcup_i \operatorname{Int} C_i$. Therefore L is locally tame at each point of G'.

At every point of $\Omega_{r-q} - G_{r-q}$, Pr^*_{r-q} is a local homeomorphism. Therefore L is everywhere locally tame in N^3. The lemma follows.

Thus Theorem 5.8 applies to N^3 and L. Let $K(N^3)$ be as in the conclusion of Theorem 5.8. Now each set $\mathrm{Pr}^*_{r-q+1}\mathrm{Bd}\ C_i$ is a projective plane and is a component of $\mathrm{Bd}\ N^3$. Therefore each such set forms a subcomplex of $K(N^3)$ in which $L \cap \mathrm{Pr}^*_{r-q+1}\mathrm{Bd}\ C_i$ forms a subcomplex. Since $K(C_i) = \mathrm{St}\ \mathrm{Pr}^*_{r-q+1}\ P_i$, it follows that $\mathrm{Pr}^*_{r-q+1}\ C_i$ has a triangulation which has all the stated properties of $K(C_i)$, such that $K(N^3) \cup \bigcup_i K'(C_i)$ is a triangulation of Ω^*_{r-q+1}. Passing to Ω_{r-q+1} as before, we get the desired triangulation $K(\Omega_{r-q+1})$. This completes the proof of Theorem 6.1.

We recall that $n = 2^q p_{q+1} p_{q+2} \cdots p_r$. For $q = 0$, the proof of Theorem 1.4 was complete at the end of §4; for $q = 1$, we have $\Omega_{r-q+1} = \Omega_r$, so that Theorem 1.4 follows from Theorem 6.1. Hereafter we suppose that $q > 1$.

From the proof of Theorem 6.1 it is evident that every point of Ω_{r-q+1} has (a) an open Euclidean neighborhood, (b) a closed neighborhood which is a 3-cell or (c) a closed neighborhood which is homeomorphic to the join of a point with a projective plane. Such a metric space will be called a *3-manifold with boundary and singularities*. For each such space N, let $\mathrm{Int}\ N$ be the set of all points which have neighborhoods of type (a); let $\mathrm{Bd}\ N$ be the set of all points of N which have neighborhoods of type (b) but not of type (a); and let $S(N)$ be the set of all points of N which have neighborhoods of type (c).

We recall that f induces a homeomorphism $f_*\colon \Omega_{r-q+1} \leftrightarrow \Omega_{r-q+1}$.

THEOREM 6.2. *Let* $P \in M$ *and let* $Q = \mathrm{Pr}_{r-q+1}P \in \Omega_{r-q+1} - \mathrm{Int}\ \Omega_{r-q+1}$. *Then* Q *has period exactly* 2^{q-1} *relative to* f_*.

PROOF. Suppose that $f_*^{2^{q-2}}(Q) = Q$. By abuse of language we use the same symbol f_* for the induced homeomorphism $\Omega_{r-q} \leftrightarrow \Omega_{r-q}$. Now $Q \notin \mathrm{Int}\ \Omega_{r-q+1}$ only if $\mathrm{Pr}_{r-q}\ P$ is a fixed point of $f_*^k = f_*^{2^{q-1}}$. Therefore $Q = \mathrm{Pr}_{r-q+1}\ P$ and $\mathrm{Pr}_{r-q}\ P$ are exactly the same set of points in M. Our assumption $f_*^{2^{q-2}}(Q) = Q$ means precisely that $f^{2^{q-2}}(Q) = Q$ (where $Q \subset M$). Now $f_*^{2^{q-2}}\colon \Omega_{r-q} \leftrightarrow \Omega_{r-q}$ either preserves or reverses local orientation at Q. In either case, $f_*^k = (f_*^{2^{q-2}})^2$ preserves local orientation at Q. Therefore $\dim G_{r-q} = 1$, and Ω_{r-q+1} is locally Euclidean at Q, which contradicts the hypothesis for Q.

Consider now what happens when we iterate this process to pass from Ω_{r-q+1} to Ω_{r-q+2}. We suppose that $K(\Omega_{r-q+1})$ is as in Theorem 6.1 and that the points Q of $S(\Omega_{r-q+1})$ have disjoint closed stars $\mathrm{St}\ Q$ in $K(\Omega_{r-q+1})$. Evidently $K(\Omega_{r-q+1})$ can be chosen so that these stars are permuted by $f_*\colon \Omega_{r-q+1} \leftrightarrow \Omega_{r-q+1}$ and hence also by $f_*^{2^{q-2}}$. Consider the set

$$N = \mathrm{Cl}\left[\Omega_{r-q+1} - \bigcup_Q |\mathrm{St}\ Q|\right].$$

As before, let G_{r-q+1} be the fixed-point set of $f_*^{2^{q-2}}$. By Theorem 6.2, $G_{r-q+1} \subset \mathrm{Int}\ N$.

Now let N' be a "doubling" of N, $= N \cup \phi(N)$, where ϕ is a homeomorphism, $\phi|\mathrm{Bd}\ N$ is the identity, and $N \cap \phi(N) = \mathrm{Bd}\ N = \mathrm{Bd}\ \phi(N)$. For each subset A of N, let A' be the doubling $A \cup \phi(A)$ of A. We define a doubling g' of the homeomorphism $g = f_*^{2^{q-2}}|N$ in the obvious way: for each $P \in N$, $g'(P) = g(P)$

and $g'(\phi(P)) = \phi(g(P))$. Then the fixed-point set of g' is G'_{r-q+1}; $G'_{r-q+1} \cap \operatorname{Bd} N = \varnothing$; and by Theorem 3.1, G'_{r-q+1} is a manifold. Let G be a component of G_{r-q+1}. If G is 1-dimensional, then it follows as in the proof of Lemma 6.1.1 that $G = \operatorname{Pr}_{r-q+1} F$, where F is a component of $F_{n/2}$ and a 1-sphere. As before, it follows that the union of the 1-dimensional components of G_{r-q+1} is tame. So also is the union of the 2-dimensional components. Therefore G'_{r-q+1} is tame in N'. Now the sets $N \cap \operatorname{Pr}_{r-q+1} F_{n_s}$ are tame in N (under our conditions for $K(\Omega_{r-q+1})$). From this it follows that the set

$$\operatorname{Bd} N \cup \bigcup_s (N \cap \operatorname{Pr}_{r-q+1} F_{n_s})'$$

is tame in N'; the point is that $\operatorname{Bd} N$ intersects a set $(N \cap \operatorname{Pr}_{r-q+1} F_{n_s})'$ only where the former is pierced by a 1-dimensional component of the latter.

Thus, in forming a triangulation of the orbit space Ω of g', we are in essentially the same situation as in the proof of Theorem 6.1, except that we need to treat the invariant set $\operatorname{Bd} N$ as if it were part of the invariant set $\bigcup_s (N \cap \operatorname{Pr}_{r-q+1} F_{n_s})'$. Let Pr be the projection $N' \to \Omega$. By the same methods as in the proof of Theorem 6.1, we get a triangulation $K(\Omega)$ in which (a) each of the sets $\operatorname{Pr} \operatorname{Bd} N$, $\operatorname{Pr}(N \cap \operatorname{Pr}_{r-q+1} F_{n_s})'$ forms a subcomplex and (b) each simplex intersects each set mentioned in (a) in a simplex. Let Ω_g be the orbit space of $g = f_*^{2^{q-2}}|N$ and let Pr_g be the projection $N \to \Omega_g$. We now have a triangulation $K(\Omega_g)$ in which each set $\operatorname{Pr}_g(N \cap \operatorname{Pr}_{r-q+1} F_{n_s})$ forms a subcomplex. Now some components of $\operatorname{Bd} \Omega_g$ may be of the type $\operatorname{Pr} M^2$, where M^2 is a projective plane in Ω_{r-q+1} and a set $|\operatorname{St} Q|$ is the join of M^2 with Q. Now each such set $|\operatorname{St} Q|$ has a join-structure in which the sets $|\operatorname{St} Q| \cap \operatorname{Pr}_{r-q+1} F_{n_s}$ appear also as joins with Q (unless they consist of Q alone). Therefore we can triangulate the sets $\operatorname{Pr}^*_{r-q+2}|\operatorname{St} Q| \subset \Omega_{r-q+2}$ by forming the join of $\operatorname{Pr}^*_{r-q+2} Q$ with the triangulation of $\operatorname{Pr} M^2$ that is already given. (Here we are regarding Ω^*_{r-q+2} and Ω_{r-q+2} as indistinguishable.)

Thus–possibly after a subdivision–we get a triangulation $K(\Omega_{r-q+2})$ which satisfies the conditions of Theorem 6.1. Iterations of this process preserve its preconditions; and so, in a finite number of steps, we get the desired triangulation $K(\Omega_r)$. Theorem 1.4 follows.

7. Proofs of Theorems 1.5 and 1.6. Let f: $\mathbf{S}^3 \leftrightarrow \mathbf{S}^3$, n, and F be as in Theorem 1.5. We shall use the notations $p_1 p_2 \cdots p_r$, F_i, and F_{n_i} as in Theorem 1.1 and its proof. By a classic result of P. A. Smith [$\mathbf{S}_1$, Theorem 4, p. 707], F_{n_i} is a sphere of some dimension; and since $F \subset F_{n_i}$, it follows that F_{n_i} is either a 1-sphere or a 2-sphere.

LEMMA 7.1. *For each* i, $F_{n_i} = F$.

PROOF. If F_{n_i} is a 1-sphere, this is clear. If not, F_{n_i} is a 2-sphere and we have $f(F_{n_i}) = F_{n_i}$; the proof is exactly the same as that of Theorem 3.8. Thus $f|F_{n_i}$ is a periodic homeomorphism of a 2-sphere onto itself with a 1-sphere F as its fixed-point set. Therefore $f|F_{n_i}$ is an involution and n_i is even, $= 2k$ for some positive integer k. By Theorem 3.3 it follows that f^{n_i}: $\mathbf{S}^3 \leftrightarrow \mathbf{S}^3$ preserves local orientation at each point of F_{n_i}. Since $\dim F_{n_i} = 2$, this contradicts Theorem 3.4. Therefore F_{n_i} is a 1-sphere and $F_{n_i} = F$, as desired.

Since F is tame, it follows that f is weakly statically tame. Thus Theorem 1.5 follows from Theorem 1.1.

Now let f: $S^3 \leftrightarrow S^3$ and F be as in Theorem 1.6. By Theorem 1.5, S^3 has a triangulation $K(S^3)$ relative to which f is simplicial. By Theorem 1.4 of **[M]**, F is the boundary of a 2-cell which is polyhedral relative to $K(S^3)$. Now $K(S^3)$ can be chosen so that for each $\sigma \in K(S^3)$, $f(\sigma) = \sigma$ only if $f|\sigma$ is the identity. Let v be a vertex of $K(S^3)$, lying in F. Then Bd|St v| is a 2-sphere S^2, and $f|S^2$ is periodic, with exactly two fixed points. Since f is simplicial, it follows easily that $f|S^2$ preserves orientation. (We hardly need Kerékjártó's results in such a case.) Therefore f preserves orientation. The main result of **[M$_9$]** asserts that under all these conditions, f is conjugate to a rotation, which was to be proved. (For a simpler proof of the result of **[M$_9$]**, see P. A. Smith **[S$_2$]**.)

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